Canonical forms for pseudo-Riemannian projectively equivalent metrics: Jordan block case

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Main definition and equation

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$$L = \left(\frac{\det \bar{g}}{\det g} \right)^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

L is (pseudo) self-adjoint w.r.t. both g and $\bar{g}$. Notice: $\bar{g} = \frac{1}{\det L} g L^{-1}$.

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Theorem

g and $\bar{g}$ are projectively equivalent if and only if L satisfies the following equation:

$$\nabla_u L = \frac{1}{2}\left(u \otimes d\ \mathrm{tr} L + (u \otimes d\ \mathrm{tr} L)^*\right)$$

for any vector field u.

Jordan block case

Assume that L is similar to a Jordan block with eigenvalue λ and $\lambda \neq \mathrm{const}$.
Linear algebra (Jordan-Kronecker theorem):

Lemma

At each tangent space $T_x M$ there is a basis $e_1, \dots, e_n$ such that

$$g = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}, \qquad L = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

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Important: *These vectors are uniquely defined (up to sign).*

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Standard decomposition: $L = \lambda \cdot \mathrm{Id} + N$, where N is nilpotent.

Main equation:

$$\nabla_u N = \frac{n}{2}(u \otimes d\lambda + (u \otimes d\lambda)^*) - d\lambda(u) \cdot \mathrm{Id},$$

where $u \otimes d\lambda$ with respect to g, in our case, means the transpose of $u \otimes d\lambda$ with respect to the other diagonal.

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Since the Nijenhuis tensor of L vanishes, we have $d\lambda(e_i) = 0$ for $i = 1, \dots, n-1$, and $d\lambda(e_n) = \lambda'$ so that $d\lambda = (0, \dots, 0, \lambda')$.

Using the explicit form of $d\lambda$:

$$\nabla_u N = \lambda' \begin{pmatrix} \frac{n-2}{2}u_n & \frac{n}{2}u_{n-1} & \frac{n}{2}u_{n-2} & \dots & \frac{n}{2}u_2 & nu_1 \\ & -u_n & & & & \frac{n}{2}u_2 \\ & & -u_n & & & \frac{n}{2}u_3 \\ & & & \ddots & & \\ & & & & -u_n & \frac{n}{2}u_{n-1} \\ & & & & & \frac{n-2}{2}u_n \end{pmatrix}$$

Our goal is to find $\nabla_{e_i} e_j$ and $[e_i, e_j]$ from this formula.

For any vector u, consider a linear operator B_u defined on our basic vector fields as follows $B_u(e_i) = \nabla_u e_i$.
Since N is "constant" with respect to our basis, we can rewrite the standard Leibniz rule

$$\nabla_u(Nv) = (\nabla_u N)v + N\nabla_u v$$

as

$$\nabla_u N = B_u N - NB_u.$$

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Linear Algebra again:
the solution exists and is unique (because N is regular!)

Here is the answer:

$$
B_u = \lambda' \begin{pmatrix}
\frac{n}{2} u_{n-1} & \frac{n}{2} u_{n-2} & \cdots & \frac{n}{2} u_2 & \frac{n}{2} u_1 & 0 \\
(1 - \frac{n}{2}) u_n & & & & & -\frac{n}{2} u_1 \\
& (2 - \frac{n}{2}) u_n & & & & -\frac{n}{2} u_2 \\
& & \ddots & & & \vdots \\
& & & (\frac{n}{2} - 2) u_n & & -\frac{n}{2} u_{n-2} \\
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B_u contains all the information about covariant derivatives $\nabla_{e_i} e_j$ and, consequently, about $[e_i, e_j]$ (since $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$).

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Conclusion:

The vector fields $e_1, \dots, e_{n-1}$ commute. And

$$[e_i, e_n] = -i\lambda' e_{i+1}.$$

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There exists a regular coordinate system such that:

$$\partial_{x_1} = e_1, \dots, \partial_{x_{n-1}} = e_{n-1}, \partial_{x_n} = \sum_{i=1}^{n} a_i e_i,$$

where

$$\left(\begin{array}{l} a_1 = 0, \\ a_2 = \lambda'_{x_n} x_1, \\ a_3 = 2\lambda'_{x_n} x_2, \\ \dots \\ a_{n-1} = (n-2)\lambda'_{x_n} x_{n-2}, \\ a_n = (n-1)\lambda'_{x_n} x_{n-1} + 1, \end{array}\right)$$

and λ is an arbitrary function of x_n.

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Now to finish the process, we just rewrite L and g in these "canonical coordinates". Again by using our favorite Linear Algebra:

$$L \to P^{-1} L P \quad \text{and} \quad g \to P^{\top} g P$$

Theorem

In the case of a single Jordan block, there exist a local coordinate system $(x_1, \dots, x_n)$ in which g and L take the following canonical form:

$$g = \begin{pmatrix} & & & & a_n \\ & & & 1 & a_{n-1} \\ & & \iddots & & \vdots \\ & 1 & & & a_2 \\ a_n & a_{n-1} & \dots & a_2 & \sum_{i=1}^n a_i a_{n-i+1} \end{pmatrix}$$

and

$$L = \begin{pmatrix} \lambda(x_n) & 1 & & & a_2 \\ & \lambda(x_n) & 1 & & a_3 \\ & & \ddots & & \\ & & & \lambda(x_n) & a_n \\ & & & & \lambda(x_n) \end{pmatrix}$$

where $\lambda = \lambda(x_n)$ is an arbitrary function of x_n and
$a_k = (k-1)\lambda'_{x_n} x_{k-1} \quad (k \neq n), \qquad a_n = (n-1)\lambda'_{x_n} x_{n-1} + 1.$