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Algebraic and Geometric Properties of Quadratic Hamiltonians Determined by Sectional Operators

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Abstract—Following the terminology introduced by V. V. Trofimov and A. T. Fomenko, we say that a self-adjoint operator $\phi\colon \mathfrak{g}^* \to \mathfrak{g}$ is *sectional* if it satisfies the identity $\operatorname{ad}^*_{\phi x} a = \operatorname{ad}^*_{\beta} x$, $x \in \mathfrak{g}^*$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $a \in \mathfrak{g}^*$ and $\beta \in \mathfrak{g}$ are fixed elements. In the case of a semisimple Lie algebra $\mathfrak{g}$, the above identity takes the form $[\phi x, a] = [\beta, x]$ and naturally arises in the theory of integrable systems and differential geometry (namely, in the dynamics of n-dimensional rigid bodies, the argument shift method, and the classification of projectively equivalent Riemannian metrics). This paper studies general properties of sectional operators, in particular, integrability and the bi-Hamiltonian property for the corresponding Euler equation $\dot{x} = \operatorname{ad}^*_{\phi x} x$.

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1. INTRODUCTION

The term "sectional operator" was suggested by Trofimov and Fomenko [1] for explicitly describing a special class of operators determining dynamical systems with many first integrals. Operators of this type first appeared in Manakov's paper [2], in which it was shown that the map $\phi\colon so(n) \to so(n)$ defined by $(\phi X)_{ij} = (b_i - b_j)/(a_i - a_j)X_{ij}$, i.e., satisfying the identity

$$[\phi X, A] = [X, B], \qquad X = (X_{ij}) \in so(n), \tag{1.1}$$

where A and B are diagonal matrices, has the following remarkable property: the Euler equations $\dot{X} = [\phi X, X]$, which generalize the classical equations of the dynamics of a rigid body to the n-dimensional case, admit a Lax representation with a spectral parameter and, therefore, can be integrated in θ-functions.

Mishchenko and Fomenko [3]–[7] proposed a general construction, called the argument shift method, which yields completely integrable systems similar to those considered by Manakov on any semisimple Lie algebra $\mathfrak{g}$. The quadratic Hamiltonians $H(x) = \langle \phi x, x\rangle/2$ of such systems were determined by a series of self-adjoint operators $\phi\colon \mathfrak{g} \to \mathfrak{g}$, whose basic algebraic property was, as previously, an identity of the form

$$[\phi x, a] = [x, b], \qquad a, b \in \mathfrak{h}, \tag{1.2}$$

where $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$ and $a \in \mathfrak{h}$ is a regular element. In [5], a convenient explicit formula for ϕ in terms of the natural orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ of the Lie algebra $\mathfrak{g}$ into a direct sum of subspaces was obtained, namely,

$$\phi(x) = \phi_{a,b,D} x = \operatorname{ad}_a^{-1} \operatorname{ad}_b x_1 + D x_2, \tag{1.3}$$

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where $x = x_1 + x_2$, $x_1 \in \mathfrak{h}$, $x_2 \in \mathfrak{h}^\perp$, $\mathrm{ad}_a^{-1}\colon \mathfrak{h}^\perp \to \mathfrak{h}^\perp$, and $D\colon \mathfrak{h} \to \mathfrak{h}$ is any self-adjoint operator.

In subsequent works (see [1], [8], [9]), Fomenko and Trofimov constructed analogs of the operators $\phi_{a,b,D}$ on any symmetric space; these operators turned out to be closely related to the tensor curvatures of these spaces.

In turn, Vinberg succeeded in "lifting" the quadratic Hamiltonians determined by sectional operators to the universal enveloping algebra [10] and obtained a quantum analog of the commuting Hamiltonians $H(x)$, and Meshcheryakov [11] showed that identity (1.2) is a characteristic property of those operators $\phi\colon \mathfrak{g} \to \mathfrak{g}$ for which the corresponding Euler equations $\dot{x} = [\phi x, x]$ are Hamiltonian with respect to the pencil of brackets $\mu\{\,\cdot\,,\,\cdot\,\} + \lambda\{\,\cdot\,,\,\cdot\,\}_a$ (see below).

Finally, it has turned out quite recently that identity (1.1) unexpectedly arises in completely different domains of geometry. In [12], it was shown that this identity is satisfied by the curvature tensors on Riemannian manifolds admitting nontrivial projectively equivalent metrics; moreover, this identity leads to some remarkable geometric properties of such manifolds.[1]

Our objective in this paper is to study general properties of sectional operators determined by identities of type (1.2) in the case of any Lie algebra $\mathfrak{g}$ and any parameters a and b: we establish the existence and give an explicit description of sectional operators, and also prove the integrability and the property of being bi-Hamiltonian of the corresponding Euler equation. We emphasize that a natural reformulation of (1.2) is related to the coadjoint, rather than adjoint, representation of the algebra, because the Euler equation is written on the coalgebra and carries over to the algebra only in the presence of a nondegenerate invariant form. Such a setting is quite natural, and its generalization to the nonsemisimple case has already been performed in [9], [13] under certain additional constraints on the parameters a and b. Here we impose no a priori constraints on these parameters. This is determined, to a large degree, by the fact that singular parameter values are of most interest in the problems of Riemannian geometry mentioned above.

All considerations are performed over the field $\mathbb{C}$ of complex numbers, but most of the proofs apply to any field of characteristic zero.

2. DEFINITION, EXISTENCE THEOREM, AND EXPLICIT FORMULA FOR SECTIONAL OPERATORS

Let $\mathfrak{g}$ be any finite-dimensional Lie algebra, and let $\mathfrak{g}^*$ be its dual space. We denote their elements by lowercase Greek and Latin letters, respectively, and the pairing between elements of the algebra and the coalgebra, by the brackets $\langle\,\cdot\,,\,\cdot\,\rangle$. It is convenient for our purposes to treat bilinear forms on $\mathfrak{g}$ (on $\mathfrak{g}^*$) as operators from $\mathfrak{g}$ to $\mathfrak{g}^*$ (respectively, from $\mathfrak{g}^*$ to $\mathfrak{g}$). By abuse of language, we say that such operators are self-adjoint when they are self-adjoint with respect to the pairing, i.e., the corresponding bilinear forms are symmetric.

Definition 1. A *sectional operator* is a self-adjoint operator $\phi\colon \mathfrak{g}^* \to \mathfrak{g}$ satisfying the identity

$$\mathrm{ad}^*_{\phi x}\, a = \mathrm{ad}^*_\beta\, x, \qquad x \in \mathfrak{g}^*, \tag{2.1}$$

for some fixed $\beta \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$.

In this case, we talk about a *sectional operator with parameters a and β.*

It is easy to see that identity (2.1) can be rewritten in one of the equivalent forms

$$\langle a, [\phi x, \nu]\rangle = \langle x, [\beta, \nu]\rangle \qquad \text{for any} \quad \nu \in \mathfrak{g}, \quad x \in \mathfrak{g}^* \tag{2.2}$$

and

$$\phi(\mathrm{ad}^*_\nu\, a) = -[\beta, \nu] \qquad \text{for any} \quad \nu \in \mathfrak{g}. \tag{2.3}$$

Clearly, in the semisimple case, the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form transforms identity (2.1) precisely into (1.2) (in which the notation β is changed to b), provided that $ad_x y = [y, x]$.

[1] Certainly, the term "sectional" was in no way related to curvature tensors; however, interestingly, in the situation described above, the sectional operator ϕ associated with the curvature tensor describes precisely the sectional curvature of the manifold.

In this definition, we impose no a priori constraints on a and β; thus, the natural question of the existence of operators with given parameters arises. Let $\operatorname{Ann} a = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi a = 0\}$ denote the stationary subalgebra of a point $a \in \mathfrak{g}^*$ with respect to the coadjoint action (we call it the *annihilator* of this point). We also define two more subalgebras:

1) the subalgebra
$$\mathfrak{b}_a = \{\xi \in \mathfrak{g} \mid \langle \operatorname{ad}^*_\xi a, \eta\rangle = 0 \text{ for all } \eta \in \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]\}$$
containing the annihilator of a (the fact that $\mathfrak{b}_a$ is indeed a subalgebra follows at once from the fact that the commutator subgroup $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$);

2) the subalgebra $\mathfrak{g}^{\operatorname{Ann} a} = \{\xi \in \mathfrak{g} \mid [\xi, \operatorname{Ann} a] = 0\}$, which is the centralizer of the annihilator $\operatorname{Ann} a$.

To each element $a \in \mathfrak{g}^*$ we assign the subalgebra $\mathfrak{g}_a = \mathfrak{b}_a \cap \mathfrak{g}^{\operatorname{Ann} a}$, which is the intersection of these two subalgebras.

Theorem 1. *A necessary and sufficient condition for the existence of a sectional operator ϕ with given parameters $a \in \mathfrak{g}^*$ and $\beta \in \mathfrak{g}$ is $\beta \in \mathfrak{g}_a$.*

Proof. We use identity (2.3), which is equivalent to the definition of a sectional operator. This identity at once determines the operator ϕ on the set of covectors of the form $y = \operatorname{ad}^*_\nu a$, which, obviously, coincides with the tangent space to the orbit of the coadjoint representation passing through a:
$$T_a\mathscr{O}(a) = \{y \in \mathfrak{g}^* \mid y = \operatorname{ad}^*_\nu a \text{ for some } \nu \in \mathfrak{g}\}.$$
Namely, according to (2.3), for any $y \in T_a\mathscr{O}(a)$, we simply set $\phi(y) = -[\beta, \nu]$, where ν is a vector satisfying the condition $y = \operatorname{ad}^*_\nu a$. Since the vector ν is determined modulo $\operatorname{Ann} a$, it follows that $\phi(y)$ is well defined if and only if $[\beta, \operatorname{Ann} a] = 0$, i.e., $\beta \in \mathfrak{g}^{\operatorname{Ann} a}$.

Thus, we have the subspace $T_a\mathscr{O}(a) \subset \mathfrak{g}^*$ on which the required operator is already defined. The question is whether this operator can be extended to the entire space so that the obtained operator is self-adjoint.

Clearly, this can be done if and only if
$$\langle \phi(y), z\rangle = \langle y, \phi(z)\rangle \tag{2.4}$$
for any two vectors $y, z \in T_a\mathscr{O}(a)$.

In matrix language, the question is as follows. Given a matrix of the form
$$\begin{pmatrix} A_1 & * \\ A_2 & * \end{pmatrix}, \tag{2.5}$$
where A_1 denotes a diagonal block and the asterisks stand for indeterminate components, is it possible to replace the asterisks by numbers so that the matrix becomes symmetric? The answer is obvious: this can be done if and only if the diagonal block A_1 is symmetric, which is precisely equivalent to (2.4).

Setting $y = \operatorname{ad}^*_\eta a$ and $z = \operatorname{ad}^*_\zeta a$, we can rewrite (2.4) in the form
$$\langle \phi(\operatorname{ad}^*_\eta a), \operatorname{ad}^*_\zeta a\rangle = \langle \operatorname{ad}^*_\eta a, \phi(\operatorname{ad}^*_\zeta a)\rangle,$$
or, again applying (2.3), in the form
$$\langle [\eta, \beta], \operatorname{ad}^*_\zeta a\rangle = \langle \operatorname{ad}^*_\eta a, [\zeta, \beta]\rangle;$$
this is equivalent to the relation
$$\begin{aligned} 0 &= \langle [\eta, \beta], \operatorname{ad}^*_\zeta a\rangle - \langle \operatorname{ad}^*_\eta a, [\zeta, \beta]\rangle \\ &= \langle a, [[\eta, \beta], \zeta] - [[\zeta, \beta], \eta]\rangle = \langle a, [[\zeta, \eta], \beta]\rangle = \langle \operatorname{ad}^*_\beta a, [\zeta, \eta]\rangle, \end{aligned}$$
which means precisely that $\beta \in \mathfrak{b}_a$.

Thus, for the existence of a sectional operator, the simultaneous fulfillment of the two conditions $\beta \in \mathfrak{g}^{\operatorname{Ann} a}$ and $\beta \in \mathfrak{b}_a$ is necessary and sufficient, as required. $\square$

The proof of the theorem gives a fairly natural explicit formula for a sectional operator ϕ. To make it transparent, we again use the matrix interpretation. The symmetric matrix obtained from (2.5) by reconstructing the missing components has the form

$$\begin{pmatrix} A_1 & A_2^\top \\ A_2 & A_3 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_2^\top \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_3 \end{pmatrix}, \tag{2.6}$$

where A_3 is any symmetric matrix.

Let us do the same in the particular case under consideration. Choose any subspace $N \subset \mathfrak{g}^*$ complementary to $T_a\mathscr{O}(a)$ (from the geometric point of view, N is a transversal space to the orbit of the coadjoint representation at the point $a \in \mathfrak{g}^*$, and its choice is, of course, nonunique). Then we have the direct sum decomposition $\mathfrak{g}^* = T_a\mathscr{O}(a) \oplus N$. In the Lie algebra $\mathfrak{g}$, the dual decomposition $\mathfrak{g} = N^\perp \oplus \operatorname{Ann} a$ automatically arises.

Let $\mathscr{A}_a\colon \mathfrak{g} \to T_a\mathscr{O}(a)$ be the linear operator defined by $\mathscr{A}_a(\xi) = \operatorname{ad}^*_\xi a$. Since its kernel coincides with $\operatorname{Ann}(a)$, if follows that the inverse map

$$\mathscr{A}_a^{-1}\colon T_a\mathscr{O}(a) \to N^\perp$$

is well defined.

In this notation, the part of the sectional operator rigidly determined by identity (2.3) is written in the form $-\operatorname{ad}_\beta \mathscr{A}_a^{-1}\pi$ (in the matrix notation used above, this is the first, known, column), where $\pi\colon \mathfrak{g}^* \to T_a\mathscr{O}(a)$ is the natural projection associated with the decomposition fixed above.

Lemma 1. *The operator adjoint to* $-\operatorname{ad}_\beta \mathscr{A}_a^{-1}\pi$ *has the form* $\mathscr{A}_a^{-1}\operatorname{ad}^*_\beta$.

Proof. The condition $\beta \in \operatorname{Ann} a$ implies $\operatorname{ad}^*_\beta y \in T_a\mathscr{O}_a$ for any $y \in \mathfrak{g}^*$; therefore, the expression $\mathscr{A}_a^{-1}\operatorname{ad}^*_\beta$ makes sense. Setting $\mathscr{A}_a^{-1}(\pi x) = \xi$ and $\mathscr{A}_a^{-1}\operatorname{ad}^*_\beta y = \eta$, we obtain

$$\begin{aligned}\langle -\operatorname{ad}_\beta \mathscr{A}_a^{-1}\pi x, y\rangle &= \langle -\operatorname{ad}_\beta \xi, y\rangle = \langle \xi, \operatorname{ad}^*_\beta y\rangle = \langle \xi, \operatorname{ad}^*_\eta a\rangle \\ &= -\langle \eta, \operatorname{ad}^*_\xi a\rangle = \langle \pi x, A_a^{-1}\operatorname{ad}^*_\beta y\rangle = \langle x, A_a^{-1}\operatorname{ad}^*_\beta y\rangle,\end{aligned}$$

i.e., $(-\operatorname{ad}_\beta \mathscr{A}_a^{-1}\pi)^* = \mathscr{A}_a^{-1}\operatorname{ad}^*_\beta$. □

Thereby, the second matrix on the right-hand side of (2.6) represents the operator $\mathscr{A}_a^{-1}\operatorname{ad}^*_\beta$, which is applied only to elements from N and vanishes on $T_a\mathscr{O}(a)$. Thus, the following theorem is valid.

Theorem 2. *Let* $x = x_1 + x_2 \in \mathfrak{g}^*$ *be a decomposition of any element* x *in which* $x_1 \in T_aO(a)$ *and* $x_2 \in N$. *Then*

$$\phi(x) = -\operatorname{ad}_b \mathscr{A}_a^{-1}x_1 + \mathscr{A}_a^{-1}\operatorname{ad}^*_\beta x_2 + Dx,$$

where D *is any self-adjoint operator with image contained in* $\operatorname{Ann}(a)$.

Note that the kernel of D automatically coincides with $T_aO(a)$, so that $Dx = Dx_2$. In particular, D can be written as $D = \widetilde{D} \circ \mathsf{pr}$, where $\widetilde{D}\colon (\operatorname{Ann} a)^* \to \operatorname{Ann} a$ is any self-adjoint operator and

$$\mathsf{pr}\colon \mathfrak{g}^* \to \operatorname{Ann}^* a$$

is the natural projection. In other words, the sectional operator ϕ with given parameters a and β is determined up to a self-adjoint operator

$$\widetilde{D}\colon (\operatorname{Ann} a)^* \to \operatorname{Ann} a.$$

This fact, however, follows at once from the definition.

Remark 1. Introducing the notation

$$B = -\operatorname{ad}_\beta \mathscr{A}_a^{-1}\colon T_a\mathscr{O}(a) \to \mathfrak{g} \qquad \text{and} \qquad C = \mathscr{A}_a^{-1}\operatorname{ad}_\beta^*\colon \mathfrak{g}^* \to N^\perp$$

and taking into account the relation $(B\pi)^* = C$, we can also rewrite the formula for ϕ in the following forms:

(a) $\phi = B\pi + C(\mathrm{id} - \pi) + D$;

(b) $\phi = C + (\mathrm{id} - \pi^*)B\pi + D$;

(c) $\phi = B\pi + (B\pi)^* - \pi^* B\pi + D$;

(d) $\phi = C + C^* - \pi^* C + D$.

The formula closest to the Mishchenko–Fomenko formula (1.3) is (b): the terms C and D have the same form as the two terms in the Mishchenko–Fomenko formula, but there arises an additional term, which is necessary for the operator to become self-adjoint.

The following several examples help to clarify the structure of the algebras $\mathfrak{g}_a$ and $\mathfrak{b}_a$.

Example 1. Suppose that the Lie algebra $\mathfrak{g}$ has a nondegenerate invariant bilinear form (e.g., if $\mathfrak{g}$ is semisimple, then this is simply the Killing form). Using this form, we can identify the algebra and the coalgebra. Under this identification, $\operatorname{Ann} a$ coincides with the centralizer of the element $a \in \mathfrak{g} \simeq \mathfrak{g}^*$. This readily implies $\mathfrak{g}^{\operatorname{Ann} a} = \mathfrak{z}(\operatorname{Ann} a)$, i.e., $\mathfrak{g}^{\operatorname{Ann} a}$ is the center of the centralizer. Taking into account the fact that $\mathfrak{b}_a$ contains $\operatorname{Ann} a$, we see that, in the case under consideration, $\mathfrak{g}_a = \mathfrak{z}(\operatorname{Ann} a)$.

Example 2. Suppose that the commutator subgroup of the Lie algebra $\mathfrak{g}$ coincides with the entire algebra. Such algebras include, e.g., the semidirect sums $\mathfrak{g} = \mathfrak{k} +_\rho V$, where $\rho\colon \mathfrak{k} \to gl(V)$ is a linear representation semisimple algebra Lie $\mathfrak{k}$ (containing no trivial components). The coincidence of the commutator subgroup with the entire algebra at once implies $\mathfrak{b}_a = \operatorname{Ann} a$; therefore, $\mathfrak{g}_a = \mathfrak{z}(\operatorname{Ann} a)$, which coincides with the center of the annihilator, as well as in the preceding example.

Example 3. Let $\mathfrak{g}$ be the four-dimensional Lie algebra $A_{4,7}$ (the notation is borrowed from [14]; see also [15]) defined in the basis e_1, e_2, e_3, and e_4 by the commutation relations

$$[e_2, e_3] = e_1, \qquad [e_1, e_4] = 2e_1, \qquad [e_2, e_4] = e_2, \qquad [e_3, e_4] = e_2 + e_3.$$

The remaining commutators are set to zero.

Consider the element $a \in \mathfrak{g}^*$ having the coordinates $a = (0, 0, 1, 0)$ in the dual basis $\{e^i\}$, i.e., $a = e^3$. It is easy to see that the commutator subgroup $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is spanned by e_1, e_2, and e_3 and is a maximal isotropic subspace with respect to the 2-form $\mathscr{A}_a = (c_{ij}^k a_k)$, i.e., $\mathfrak{b}_a = \mathfrak{g}'$. In turn, $\operatorname{Ann} a$ is the commutative subalgebra spanned by e_1 and e_2 and it is maximal. Therefore,

$$\mathfrak{g}^{\operatorname{Ann} a} = \operatorname{Ann} a = \mathfrak{g}_a.$$

Example 4. Now, suppose that $\mathfrak{g}$ is the four-dimensional Lie algebra $A_{4,9}$ (the notation is borrowed from [13], [14]) defined in the basis e_1, e_2, e_3, e_4 by the commutation relations

$$[e_2, e_3] = e_1, \qquad [e_1, e_4] = e_1, \qquad [e_2, e_4] = e_2.$$

The remaining commutators are set to zero.

Let $a = (1, 0, 0, 0)$ in the dual basis, i.e., $a = e^1$. The commutator subgroup in this case is spanned by e_1 and e_2; this is again a maximal isotropic subspace for the form $\mathscr{A}_a = (c_{ij}^k a_k)$. Therefore, $\mathfrak{b}_a = \mathfrak{g}'$. Since $\operatorname{Ann} a$ consists only of zero in the case under consideration, it follows that $\mathfrak{g}^{\operatorname{Ann} a} = \mathfrak{g}$ and $\mathfrak{g}_a$ coincides with $\mathfrak{b}_a$.

In the semisimple case, sectional operators determine a left-invariant metric on the Lie group (see [3], [5]). In the general case, they determine quadratic (in momenta) left-invariant Hamiltonians on the cotangent bundle to the group. In relation to this interpretation, it is interesting to learn whether there exist nondegenerate and positive definite (in the real case) sectional operators.

Proposition 1. *If* $\mathfrak{g}^{\operatorname{Ann} a} \not\subseteq \operatorname{Ann} a$*, then the sectional operator* ϕ *is degenerate.*

Proof. Consider the centralizer $\mathfrak{g}^\beta$ of an element $\beta \in \mathfrak{g}$. Let us show that the strict inclusion $\operatorname{Ann} a \subset \mathfrak{g}^\beta$ holds. First, $\operatorname{Ann} a$ itself is contained in $\mathfrak{g}^\beta$, because, as we know, $[\beta, \operatorname{Ann} a] = 0$. Further, there are two possible cases:

1) $\beta \notin \operatorname{Ann} a$; in this case, in addition to $\operatorname{Ann} a$, the subalgebra $\mathfrak{g}^\beta$ contains the element β itself and, therefore, $\mathfrak{g}^\beta$ is strictly larger than $\operatorname{Ann} a$;

2) $\beta \in \operatorname{Ann} a$; in this case, $\mathfrak{g}^\beta$ contains the entire centralizer $\mathfrak{g}^{\operatorname{Ann} a}$, which is not contained in $\operatorname{Ann} a$ by assumption; thus, $\mathfrak{g}^\beta$ again contains $\operatorname{Ann} a$ as a proper subspace.

It follows that

$$\dim \ker \operatorname{ad}^*_\beta = \dim \mathfrak{g}^\beta > \dim \operatorname{Ann} a.$$

On the other hand, relation (2.1) directly implies that the sectional operator ϕ takes $\ker \operatorname{ad}^*_\beta$ to $\operatorname{Ann} a$. The estimate for dimensions shows that the kernel of ϕ is nontrivial. □

Now, consider the real case.

Proposition 2. *If* $\operatorname{Ann} a + [\mathfrak{g}, \operatorname{Ann} a] \neq \mathfrak{g}$*, then the sectional operator* ϕ *cannot be positive definite.*

Proof. Without loss of generality, we can assume that $\beta \in \operatorname{Ann} a$. Otherwise, the operator ϕ is degenerate by virtue of Proposition 1, and the required assertion holds automatically.

The assumption

$$\operatorname{Ann} a + [\mathfrak{g}, \operatorname{Ann} a] \neq \mathfrak{g}$$

means the existence of a nonzero element $y \in \mathfrak{g}^*$ which simultaneously belongs to $(\operatorname{Ann} a)^\perp = T_a\mathscr{O}(a)$ and to $[\mathfrak{g}, \operatorname{Ann} a]^\perp$. The former means that y can be is represented in the form $\operatorname{ad}^*_\nu a$ for some $\nu \in \mathfrak{g}$. Hence $\phi y = -[\beta, \nu]$, and we have

$$\langle \phi y, y\rangle = \langle -[\beta, \nu], y\rangle = 0,$$

because $[\beta, \nu] \in [\operatorname{Ann} a, \mathfrak{g}]$ and $y \in [\operatorname{Ann} a, \mathfrak{g}]^\perp$. Thus, y is a nontrivial isotropic vector and, therefore, ϕ is not positive definite. □

In conclusion of this section, we make one more important remark. In practical problems, the following natural question arises: Is a given operator ϕ sectional? For example, it is this question which arises when we want to clarify whether there exists a metric g' different from a given Riemannian metric g but having the same geodesics. A necessary condition is that the tensor curvature of the metric g be a sectional operator in the sense of the initial identity (1.1) for the algebra $so(n)$ (see [12]). It is also important to know how many different sectional representations a given operator ϕ has, i.e., how many pairs a, β satisfying identity (2.3) exist. These questions reduce to a fairly standard problem from linear algebra.

Suppose that an operator $\phi\colon \mathfrak{g}^* \to \mathfrak{g}$ is given and it is required to determine whether or not it is sectional. We emphasize that we are interested only in the case $a \neq 0$. As to β, clearly, this parameter is always determined modulo the center of the algebra, and we do not consider it in what follows. It is natural to distinguish between two cases. We say that the operator ϕ has the *trivial sectional representation with parameters* $a \neq 0$ *and* β if it satisfies identity (2.3), i.e., $\phi\mathscr{A}_a = -\operatorname{ad}_\beta$, in which both sides are trivial, i.e.,

$$\phi\mathscr{A}_a = 0 \qquad \text{and} \qquad -\operatorname{ad}_\beta = 0.$$

In the notation of Theorem 2, this means that the operator ϕ has only the trivial part D. This possibility should be borne in mind, but it is hardly interesting.

It is fairly easy to understand whether a given operator has a trivial sectional representation. It suffices to consider the map $a \mapsto \phi\mathscr{A}_a$ as a linear operator $\mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g}^*$. The operator ϕ admits a trivial sectional representation if and only if the kernel of this map is nontrivial; the dimension of the kernel determines the number of independent trivializing parameters $a \in \mathfrak{g}^*$.

Geometrically, the nontriviality of the kernel means that the image of ϕ as a subspace of $\mathfrak{g}$ is contained in the annihilator of some element $a \neq 0$ (here it is more convenient to use identity (2.1)). This is surely the case for any ϕ if the commutator subgroup $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is strictly less than the Lie algebra $\mathfrak{g}$. In this case, it is sufficient to take $a \in (\mathfrak{g}')^{\perp}$, because such elements are characterized by the property $\operatorname{Ann} a = \mathfrak{g}$, or, equivalently, $\mathscr{A}_a = 0$.

To clarify the question about nontrivial sectional representations, consider the image of the map $a \mapsto \phi\mathscr{A}_a$ as a subspace $P_1 \subset \mathfrak{g} \otimes \mathfrak{g}^*$. Similarly, as β ranges over the Lie algebra $\mathfrak{g}$, the operators of the form $-\operatorname{ad}_\beta$ constitute a subspace $P_2 \subset \mathfrak{g} \otimes \mathfrak{g}^*$. Note that P_2 is precisely the associated Lie algebra.

Both subspaces are described quite explicitly, and the answer to the questions posed above is given by the following proposition.

Proposition 3. (1) *An operator ϕ admits a trivial sectional representation if and only if*

$$\dim P_1 \leq \dim \mathfrak{g}.$$

The dimension of the space of trivializing parameters $a \in \mathfrak{g}$ is equal to $\dim \mathfrak{g} - \dim P_1$.

(2) *An operator ϕ admits a nontrivial sectional representation if and only if*

$$P_1 \cap P_2 \neq \{0\}.$$

The dimension of this intersection determines the number of independent nontrivial sectional representations for ϕ (*modulo the trivial ones*).

3. INTEGRALS AND BI-HAMILTONIAN SYSTEMS DETERMINED BY SECTIONAL OPERATORS. I: ARGUMENT SHIFT METHOD

Consider the quadratic function $f(x) = \langle \phi x, x\rangle/2$ associated with a sectional operator $\phi\colon \mathfrak{g}^* \to \mathfrak{g}$ and the corresponding Hamiltonian system

$$\dot{x} = \operatorname{ad}^*_{\phi x} x \tag{3.1}$$

on $\mathfrak{g}^*$ in the sense of the standard Poisson–Lie bracket $\{\,,\}$ on $\mathfrak{g}^*$. Recall that this Poisson bracket is defined by

$$\{f, g\}(x) = \langle x, [df, dg]\rangle = \mathscr{A}(df, dg),$$

where $\mathscr{A} = (c^k_{ij} x_k)$ is the corresponding Poisson tensor considered as a skew-symmetric form on $\mathfrak{g}$, or as an operator $\mathscr{A}\colon \mathfrak{g} \to \mathfrak{g}^*$. First, note that system (3.1) always admits at least one linear integral.

Proposition 4. *Let ϕ be a sectional operator with parameters a and β. Then the linear function $\beta(x) = \langle \beta, x\rangle$ is an integral of the Hamiltonian system* (3.1).

Proof. If ϕ is sectional, then, by using (2.2), we obtain

$$\{\beta(x), f(x)\} = \langle x, [\beta, \phi x]\rangle = \langle a, [\phi x, \phi x]\rangle = 0,$$

which proves the proposition. □

The existence of other integrals is related to the argument shift method (see [3], [5], [9], [8]) and the presence on $\mathfrak{g}^*$ of a pencil of compatible Poisson brackets $\{\,\cdot\,,\,\cdot\,\}+\lambda\{\,\cdot\,,\,\cdot\,\}_a$, where the second bracket $\{\,\cdot\,,\,\cdot\,\}_a$ is determined by the form $\mathscr{A}_a$ considered as a constant Poisson tensor, that is,

$$\{f,g\}_a(x)=\langle a,[df(x),dg(x)]\rangle=\mathscr{A}_a(df(x),dg(x)).$$

We briefly recall this construction below.

Let $I(x)$ be an invariant of the coadjoint representation of the Lie algebra $\mathfrak{g}$. Suppose that, at a point $a\in\mathfrak{g}^*$, this function can be expanded as

$$I(a+\lambda x)=f_0+\lambda f_1(x)+\lambda^2 f_2(x)+\lambda^3 f_3+\cdots. \tag{3.2}$$

Note that, in the general case, invariants are not necessarily polynomials and may even be undefined at a, but for our purposes, any *local* (and even *formal* [16]) invariant defined at a suits. Consider all smooth local invariants in a neighborhood of a and their expansions (3.2). The polynomial subalgebra $\mathscr{F}_a$ generated by the coefficients in these expansions is called the *Mishchenko–Fomenko algebra*, or the *algebra of polynomial shifts*. The main property of $\mathscr{F}_a$ is that it is commutative with respect to both brackets $\{\,\cdot\,,\,\cdot\,\}$ and $\{\,\cdot\,,\,\cdot\,\}_a$ (see [3], [5], [8], [9]). The following assertion is a natural extension of the Mishchenko–Fomenko construction to the case of any Lie algebra.

Proposition 5. *Suppose that the parameters a and β of a sectional operator ϕ satisfy the relation $\operatorname{ad}^*_\beta a=0$, i.e., $\beta\in\operatorname{Ann}(a)$. Then Eq.* (3.1) *can be written in the form*

$$\frac{d}{dt}(x+\lambda a)=\operatorname{ad}^*_{\phi x-\lambda\beta}(x+\lambda a),$$

i.e., is Hamiltonian with respect to the bracket $\{\,\cdot\,,\,\cdot\,\}+\lambda\{\,\cdot\,,\,\cdot\,\}_a$ with Hamiltonian

$$f_\lambda(x)=\frac{\langle\phi x,x\rangle}{2}-\lambda\beta(x)\qquad\text{for any}\quad\lambda\in\mathbb{R}.$$

The Mishchenko–Fomenko algebra $\mathscr{F}_a$ consists of the first integrals of these equations.

Proof. The former assertion (an analog of the Lax representation) follows directly from the definition of a sectional operator and the assumption $\beta\in\operatorname{Ann}(a)$. The latter follows from the fact that $I(a+\lambda x)$ is a Casimir function for the bracket

$$\{\,\cdot\,,\,\cdot\,\}_a+\lambda\{\,\cdot\,,\,\cdot\,\}\qquad(\lambda\neq 0)$$

with respect to which system (3.1) is Hamiltonian. Therefore, this function is a first integral for any $\lambda\neq 0$, and the same is true for all coefficients in expansion (3.2). ☐

Remark 2. The condition $\beta\in\operatorname{Ann}a$ is fairly natural and surely holds in the following cases:

1) the Lie algebra $\mathfrak{g}$ admits a nondegenerate invariant bilinear form (see Example 1);

2) the commutator subgroup $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]$ coincides with the entire algebra $\mathfrak{g}$ (see Example 2);

3) $\operatorname{Ann}a$ contains its centralizer $\mathfrak{g}^{\operatorname{Ann}a}$ (e.g., $\operatorname{Ann}a$ is a maximal commutative subalgebra).

Note that the condition $\beta\in\operatorname{Ann}a$ ensures that system (3.1) is Hamiltonian with respect to any linear combination of the form $\{\,\cdot\,,\,\cdot\,\}+\lambda\{\,\cdot\,,\,\cdot\,\}_a$; however, it says nothing about being Hamiltonian in the sense of the bracket $\{\,\cdot\,,\,\cdot\,\}_a$. In the general case, (3.1) is not necessarily Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$; however, there exists a natural class of operators related to the argument shift method for which (3.1) is Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$.

Proposition 6. *Let $f=\langle\phi x,x\rangle/2$ be a homogeneous quadratic polynomial in the Mishchenko–Fomenko algebra $\mathscr{F}_a$. Then ϕ is a sectional operator, and the corresponding equations* (3.1) *are Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$.*

Proof. The definition of the Mishchenko–Fomenko algebra readily implies that any homogeneous quadratic polynomial $f = \langle \phi x, x\rangle/2 \in \mathscr{F}_a$ can be represented as the second coefficient f_2 in expansion (3.2) for some invariant $I(x)$. Since $I(a+\lambda x)$ is a Casimir function for the bracket

$$\{\,\cdot\,,\,\cdot\,\}_a + \lambda\{\,\cdot\,,\,\cdot\,\}, \qquad \text{i.e.,} \quad (\mathscr{A}_a + \lambda\mathscr{A})dI(a+\lambda x) = 0,$$

it follows that the expansion coefficients f_i satisfy the well-known recursive relations ([3], [9])

$$\mathscr{A}_a df_1 = 0, \quad \mathscr{A}_a df_2 = -\mathscr{A}\,df_1, \quad \mathscr{A}_a df_3 = -\mathscr{A}\,df_2, \quad \dots .$$

Note that $df_1 = dI(a)$ is an element $\beta \in \mathfrak{g}$, which belongs to the annihilator $\operatorname{Ann} a$ by virtue of the first relation. The second relation is equivalent to the differential $df_2(x) = \phi x$ being a sectional operator with parameters a and $-\beta$. Finally, the third relation means precisely that, for $f_2 = f = \langle \phi x, x\rangle/2$, system (3.1) is Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$. ☐

Note that these considerations and Theorem 1 readily imply that a differential invariant of the coadjoint representation at any (not necessarily regular) point $a \in \mathfrak{g}^*$ always belongs to the center of the annihilator $\operatorname{Ann} a$. This assertion is a generalization of Kostant's theorem [17] to the case of any Lie algebra.

Thus, system (3.1) being Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$ is ensured by the inclusion

$$f = \frac{1}{2}\langle \phi x, x\rangle \in \mathscr{F}_a.$$

This automatically holds if a is regular and $\beta \in \operatorname{Ann} a$, which readily follows from the fact that $\operatorname{Ann} a$ is generated by differentials of invariants at the point a if a is regular. Thus, we have proved the following proposition.

Proposition 7. *Let $a \in \mathfrak{g}^*$ be a regular element, and let $\beta \in \operatorname{Ann} a$. Then system* (3.1) *is Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$.*

Example 5. Without the regularity assumption on a, Proposition 7 becomes false. Consider the four-dimensional Frobenius algebra from Example 3 and the element $a \in \mathfrak{g}^*$ with coordinates $a = (0,0,1,0)$ in the dual basis. The tensors $\mathscr{A}$ and $\mathscr{A}_a$ have the forms

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & 2x_1 \\ 0 & 0 & -x_1 & -x_2 \\ 0 & x_1 & 0 & -x_2 - x_3 \\ -2x_1 & x_2 & x_2 + x_3 & 0 \end{pmatrix}, \qquad \mathscr{A}_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have shown that $\mathfrak{g}_a = \operatorname{span}\{e_1, e_2\}$.

Let $\beta = e_1$. It is easy to see that the operator ϕ determined by the matrix

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is sectional, i.e., $\mathscr{A}\beta = -2x_1 e^4 = \mathscr{A}_a \phi x$. On the other hand,

$$\operatorname{ad}^*_{\phi x} x = \mathscr{A}\phi x = (0, -2x_1^2, 0, -2x_1x_3 + 2x_1x_2).$$

Since the second coordinate does not vanish, it follows that the field is not tangent to the symplectic fibers $\mathscr{A}_a$ and, hence, is not Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$.

It is natural to ask the converse question. Suppose that $g(x) = \langle \psi x, x\rangle/2$ is a quadratic Hamiltonian for which the corresponding Euler equations

$$\dot{x} = \mathrm{ad}^*_{\psi x}\, x \tag{3.3}$$

are Hamiltonian not only with respect to the standard Poisson–Lie bracket $\{\,\cdot\,,\,\cdot\,\}$ but also with respect to the constant bracket $\{\,\cdot\,,\,\cdot\,\}_a$. Is the operator ψ sectional?

Meshcheryakov showed that, in the case of a semisimple Lie algebra, the answer to this question is positive. In fact, his argument carries over word for word to the case of any Lie algebra; namely, the following theorem is valid.

Theorem 3. *Suppose that, for some self-adjoint operator ψ, the equations $\dot{x} = \mathrm{ad}^*_{\psi x}\, x$ are Hamiltonian with respect to $\{\,\cdot\,,\,\cdot\,\}_a$, i.e., there exists a function H for which*

$$\mathrm{ad}^*_{\psi x}\, x = \mathrm{ad}^*_{dH}\, a.$$

Then the map $\psi\mathscr{A}_a \colon \mathfrak{g} \to \mathfrak{g}$ is a derivation of the Lie algebra $\mathfrak{g}$. In particular, if all derivations of this Lie algebra are inner, then ψ is a sectional operator.

Proof. It follows from the compatibility of $\mathscr{A}_a$ and $\mathscr{A}$ that, for $g(x) = \langle \psi x, x\rangle/2$ and coordinate functions $\xi, \eta \in \mathfrak{g}$, we have

$$\{g, \{\xi, \eta\}\}_a + \{\eta, \{g, \xi\}\}_a + \{\xi, \{\eta, g\}\}_a + \{g, \{\xi, \eta\}_a\} + \{\eta, \{g, \xi\}_a\} + \{\xi, \{\eta, g\}_a\} = 0.$$

We know that $\mathscr{A} dg = \mathscr{A}_a dH$, i.e., $\{g,\,\cdot\,\} = \{H,\,\cdot\,\}_a$. Making the corresponding changes in the second, the third, and the fourth term, we see that their sum vanishes by virtue of the Jacobi identity for H, ξ, and η in the sense of the bracket $\{\,\cdot\,,\,\cdot\,\}_a$. Note that the first term can be rewritten in the form

$$\langle a, [\psi x, [\xi, \eta]]\rangle = -\langle \psi\, \mathrm{ad}^*_{[\xi,\eta]}\, a, x\rangle = \langle \psi\mathscr{A}_a[\xi, \eta], x\rangle.$$

Rewriting the fifth and the sixth term in a similar way and taking into account the arbitrariness x, we obtain

$$\psi\mathscr{A}_a[\xi, \eta] = [\psi\mathscr{A}_a\xi, \eta] + [\xi, \psi\mathscr{A}_a\eta].$$

Thus, $\psi\mathscr{A}_a \colon \mathfrak{g} \to \mathfrak{g}$ is a derivation. If all derivations are inner, then

$$\psi\mathscr{A}_a = -\,\mathrm{ad}_\beta \qquad \text{for some} \quad \beta \in \mathfrak{g},$$

which coincides with the definition (2.3) of a sectional operator. This completes the proof of the theorem. □

4. INTEGRALS AND BI-HAMILTONIAN SYSTEMS DETERMINED BY SECTIONAL OPERATORS. II: THE CASE OF A FROBENIUS LIE ALGEBRA

Recall that a Lie algebra $\mathfrak{g}$ is said to be *Frobenius* (see [18]) if it has index zero. This is equivalent to the requirement that the annihilator $\mathrm{Ann}\, a$ of a generic element $a \in \mathfrak{g}^*$ is trivial or, equivalently, the form $\mathscr{A}_a$ is nondegenerate and determines a (constant) symplectic structure on $\mathfrak{g}^*$.

The argument shift method does not work in this case, since the Mishchenko–Fomenko algebra is trivial because of the absence of invariants. However, the bi-Hamiltonian construction can be successfully applied in this case and leads to a large number of commuting integrals.

In this section, we assume that $a \in \mathfrak{g}^*$ is an element in general position and the Lie algebra $\mathfrak{g}$ is Frobenius. Let us see what sectional operators look like in this case.

Lemma 2. *Under the above assumptions, the subalgebra $\mathfrak{g}_a$ is nontrivial, and its dimension is equal to the codimension of the commutator subgroup $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. Since $\mathrm{Ann}\, a = \{0\}$, it follows that $\mathfrak{g}^{\mathrm{Ann}\, a} = \mathfrak{g}$. Therefore, $\mathfrak{g}_a$ coincides with the subalgebra $\mathfrak{b}_a$, which is, by definition, the "skew-orthogonal complement" of the commutator subgroup $\mathfrak{g}'$ with respect to the form $\mathscr{A}_a$. This form is nondegenerate; therefore, $\dim \mathfrak{g}_a = \mathrm{codim}\, \mathfrak{g}'$. It remains to use the fact that $\mathrm{codim}\, \mathfrak{g}' \geq 1$ for a Frobenius Lie algebra (see [18]). □

Thus, we can construct nontrivial sectional operators ϕ. In the case under consideration, they are determined by the very simple explicit formula $\phi x = \mathscr{A}_a^{-1} \operatorname{ad}^*_\beta x$ or, equivalently,

$$\phi x = -\operatorname{ad}_\beta \mathscr{A}_a^{-1}(x).$$

Below we recall a well-known general bi-Hamiltonian construction as applied to the dual space of a Frobenius Lie algebra. Consider the compatible Poisson brackets $\{\,\cdot\,,\,\cdot\,\}$ and $\{\,\cdot\,,\,\cdot\,\}_a$ on $\mathfrak{g}^*$ corresponding to Poisson tensors $\mathscr{A}$ and $\mathscr{A}_a$. Since $\mathscr{A}_a$ is nondegenerate, it follows that the recursion operator $R = \mathscr{A}\mathscr{A}_a^{-1} \colon \mathfrak{g}^* \to \mathfrak{g}^*$ is well defined. Note that, in the case under consideration, R depends linearly on $x \in \mathfrak{g}^*$ and is nondegenerate almost everywhere.

Suppose that there is a vector field v_0 which is Hamiltonian with respect to both brackets Poisson, i.e.,

$$v_0 = \mathscr{A}\, df_0 = \mathscr{A}_a df_1.$$

Then all vector fields of the form $v_k = R^k v_0$ are bi-Hamiltonian as well, i.e.,

$$v_k = \mathscr{A}\, df_k = \mathscr{A}_a df_{k-1},$$

and all functions f_k commute with each other in the sense of both Poisson structures. Note that

$$df_k = (R^*)^k df_0,$$

where $R^* = \mathscr{A}_a^{-1}\mathscr{A} \colon \mathfrak{g} \to \mathfrak{g}$ is the operator adjoint to R.

Moreover, all of these functions f_i commute with all functions of the form $g_m(x) = \operatorname{tr} R^m(x)$ (which, in turn, pairwise commute with respect to both brackets).

Now, note that relation (2.1), which defines a sectional operator ϕ, can be written in the form

$$\mathscr{A}\beta = \mathscr{A}_a \phi x \qquad \text{or} \qquad \mathscr{A}\, df_0 = \mathscr{A}_a df_1,$$

where $f_0(x) = \langle \beta, x\rangle$ is the linear function determined by the element $\beta \in \mathfrak{g}_a$ and $f_1(x) = \langle \phi x, x\rangle/2$ is the quadratic function determined by the sectional operator. Thus, we are in precisely the same situation as above and, thereby, obtain the following result.

Theorem 4. *Let $\mathfrak{g}$ be a Frobenius Lie algebra, and let ϕ be a sectional operator with parameters a and $\beta \in \mathfrak{g}_a$; suppose that $a \in \mathfrak{g}^*$ is a generic element. Then the system of Euler equations*

$$\dot x = \operatorname{ad}^*_{\phi x} x$$

is Hamiltonian with respect to the two brackets $\{\,\cdot\,,\,\cdot\,\}$ and $\{\,\cdot\,,\,\cdot\,\}_a$ and has commuting integrals of the form $g_k(x) = \operatorname{tr} R^m(x)$ and $f_k(x)$, where $f_k(x)$ is the homogeneous polynomial of degree $k+1$ uniquely determined by $df_k(x) = R^{*k}\beta$. The operator ϕ itself is determined by $\phi x = R^*\beta$.

Example 6. Consider the Lie algebra $\mathfrak{g}$ from Example 4. The pair of Poisson structures on $\mathfrak{g}^*$ has the form

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & 0 & x_1 \\ 0 & 0 & x_1 & x_2 \\ 0 & -x_1 & 0 & 0 \\ -x_1 & -x_2 & 0 & 0 \end{pmatrix}, \qquad \mathscr{A}_a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The recursion operator $R^* = \mathscr{A}_a^{-1}\mathscr{A}$ is determined by the matrix

$$R^* = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & x_1 \end{pmatrix}.$$

For β we take the element $e_1 + e_2$. Calculating $df_1(x) = \phi x = R^*\beta$, we obtain

$$f_1 = \frac{\langle \phi x, x\rangle}{2} = \frac{x_1^2}{2} + x_1 x_2.$$

The corresponding Hamiltonian system has two independent commuting integrals.

Note that the recursion operator has only one nontrivial eigenvalue. Therefore, functions of the form

$$g_m(x) = \operatorname{tr} R^m(x)$$

do not constitute a complete set of commuting functions; they must be supplemented by functions of the form f_k.

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