30 Years of Bi-Hamiltonian Systems, Bedlewo, Poland

Singular Lagrangian fibrations and bi-Hamiltonian systems

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Singular Lagrangian fibrations

We consider a symplectic manifold (M^{2n}, ω), a smooth Hamiltonian $H : M^{2n} \to \mathbb{R}$ and the corresponding Hamiltonian system

$$\frac{dx}{dt} = X_H(x) = \omega^{-1} dH(x)$$

with n commuting independent first integrals $f_1, \ldots, f_n$.
To each integrable system we can naturally assign:
1) Momentum mapping $F = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$.
2) Action of $\mathbb{R}^n$ generated by translations along $X_{f_1}, \ldots, X_{f_n}$.
3) Singular Lagrangian fibration, whose fibers are connected components of $F^{-1}(a)$, $a \in \mathbb{R}^n$; we assume that all of them are compact.

Dynamics: topological viewpoint

What do we usually want to know about a given dynamical system?
Equilibrium points, remarkable trajectories, stability, limit sets, different types of motion (depending on the initial data), bifurcations

In the case of integrable systems, all this information is contained in the "topology" of the corresponding Lagrangian fibration.

General theory and Applications:

Atiyah, Audin, Boucetta, A. Brailov, Yu. Brailov, Colin de Verdière, Cushman, Dazord, Delzant, Dufour, Duistermaat, Dullin, Eliasson, Fomenko, Guillemin, Ito, Kalashnikov, Kharlamov, Lerman, Leung, S. Matveev, V. Matveev, Miranda, Mishachev, Molino, Morozov, Nguyen Tien Zung, Orel, Oshemkov, Pogosyan, Rüssmann, Ryabov, Selivanova, Sternberg, Symington, Toulet, Umanskii, Vey, San Vu Ngoc, Zotiev, Zieschang, etc.

Question

Does the bi-Hamiltonian structure (if it exists) help to understand the qualitative properties of a system, in particular, the properties of the structure of the Lagrangian fibration?

Do bi-Hamiltonian systems possess any specific topological properties? What is the relationship between "topology" and being "bi-Hamiltonian"?

Special case of bi-Hamiltonian systems

Consider a family of degenerate compatible Poisson brackets $\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_A + \lambda\{\cdot,\cdot\}_B$ on M and assume that all of them are of the same rank $R < \dim M$.

Let $\mathcal{F}$ be the set of functions generated by the Casimir functions of all $\{\cdot,\cdot\}_\lambda$.

Classical bi-Hamiltonian fact: $\mathcal{F}$ is a commutative algebra of first integrals of any bi-Hamiltonian system associated with this pencil.

$\mathcal{F}$ generates a singular Lagrangian fibration $\mathcal{L}$ on our Poisson manifold M and we are interested in its properties.

Question. What is the singular set of this fibration, i.e., the set where the first integrals become "functionally dependent"?

Equivalently: what are those points $x \in M$ where the space in $T_x^* M$ generated by the differentials $df(x)$, $x \in \mathcal{F}$, is not maximal isotropic?

Linear algebra gives the following answer:

Proposition. x is singular if and only if there is $\lambda \in \bar{\mathbb{C}}$ such that the rank of $\{\cdot, \cdot\}_\lambda$ at x is less than R.

Principle: the nature of singularities of $\mathcal{L}$ is essentially defined by the singularities of $\{\cdot, \cdot\}_\lambda$, $\lambda \in \bar{\mathbb{C}}$.

Example 1: argument shift method

$\mathfrak{g}$ is a semisimple Lie algebra, $\{\cdot,\cdot\}$ the standard Lie-Poisson bracket

$$\{f,g\}(x) = \langle x, [df(x), dg(x)]\rangle$$

and

$$\{f,g\}_a(x) = \langle a, [df(x), dg(x)]\rangle, \qquad a \in \mathfrak{g}$$

The corresponding commuting functions (a-shifts) are $f_k^i(x)$ defined by

$$f_k(x+\lambda a) = f_k^0(x) + \lambda f_k^1(x) + \lambda^2 f_k^2(x) + \dots$$

where f_k are Casimir functions of $\mathfrak{g}$, $i < \deg f_k$.

Mischenko-Fomenko theorem: f_k^i are functionally independent, $1 \geq k \geq \operatorname{rank}\mathfrak{g}$, $0 \leq i < \deg f_k$.

Proposition. $x \in \mathfrak{g}$ is critical for the family of a-shifts (i.e., the differentials of $df_k^i(x)$ are linearly dependent), if and only if there is $\lambda \in \mathbb{C}$ such that $x + \lambda a$ is singular in the sense that the dimension of the adjoint orbit $\mathcal{O}(x + \lambda a)$ is not maximal.

In other words, the critical set K for the momentum mapping F_a (i.e., the singular set for the Lagrangian fibration) has a very simple structure: this is the cylinder over the set $\mathfrak{g}_{sing}$ of all singular adjoint orbits (more precisely the intersection of the "complex" cylinder with the "real" Lie algebra $\mathfrak{g}$):

$$K = (\mathfrak{g}_{sing} + \lambda a) \cap \mathfrak{g}, \qquad \mathfrak{g}_{sing} \subset \mathfrak{g}^{\mathbb{C}}, \lambda \in \mathbb{C}.$$

Let $\mathfrak{g}$ be a compact Lie algebra.

Question. What are generic singularities of the Lagrangian fibration associated with a-shifts? What are typical bifurcations of Lagrangian tori?

generic singularity $\Rightarrow$
rank of dF_a drops by 1 $\Rightarrow$
there is exactly one $\lambda \in \mathbb{R}$ such that $x + \lambda a$ is singular $\Rightarrow$
the singularity of $x + \lambda a$ is of $so(3)$ type $\Rightarrow$
generic singularities of $\mathcal{L}_a$ are all elliptic

Conclusions:
Typical bifurcations: $T^n \to T^{n-1} \to \emptyset$
No hyperbolic singularities
Only one family of regular tori
$F^{-1}(c)$ is connected

Example 2: Euler-Manakov top

$$\frac{d}{dt}X = [\Omega(X), X], \quad X = \Omega J + J\Omega,\ X \in so(n),\ J \text{ symmetric}$$

Integrability follows from:

$$\frac{d}{dt}(X + \lambda J^2) = [\Omega(X) + J, X + J^2]$$

Commuting first integrals: $f_{k,\lambda}(X) = \mathrm{Tr}(X + \lambda J^2)^k$

Bi-Hamiltonian structure:

$[X,Y] = [X,Y]_I = XY - YX$ and $[X,Y]_A = XAY - YAX,$

where $A = J^2$ is a symmetric matrix, I is the identity matrix.

Commuting integrals as Casimirs of $\{\cdot,\cdot\}_{A+\lambda I}$:

$$h_{k,\lambda}(X) = \mathrm{Tr}(X(A + \lambda I)^{-1})^k$$

Let $\mathcal{L}_A$ be the Lagrangian fibration on $so(n)$ generated by the family of compatible Poisson brackets $\{\cdot,\cdot\}_{A+\lambda I}$.

$K_A \subset so(n,\mathbb{R})$ singular set of $\mathcal{L}_A$

$S \subset so(n,\mathbb{C})$ set of all singular orbits (algebraic variety of codimension 3).

Proposition.

$$K_A = \mathrm{Re}\left(\cup_{\lambda\in\bar{\mathbb{C}}} (A+\lambda I)^{1/2}\, S\, (A+\lambda I)^{1/2}\right)$$

In other words, $X \in so(n)$ is a singular point for the Lagrangian fibration $\mathcal{L}_A$ if and only if X can be presented in the form

$$X = (A+\lambda I)^{1/2}\, X'\, (A+\lambda I)^{1/2}$$

where X' is a singular skew symmetric complex matrix and $\lambda \in \bar{\mathbb{C}}$.

Case n = 4:

$S \subset so(4, \mathbb{C})$ is the union of two 3-dim subspaces

$$P_1 = \begin{pmatrix} 0 & z_3 & -z_2 & z_1 \\ -z_3 & 0 & z_1 & z_2 \\ z_2 & -z_1 & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0 \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} 0 & -z_3 & z_2 & z_1 \\ z_3 & 0 & -z_1 & z_2 \\ -z_2 & z_1 & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0 \end{pmatrix}$$

The singular set for $\mathcal{L}_A$ is:

$$K_A = \bigcup_{i=1,2,\, \lambda \in \bar{\mathbb{C}}} P_i^\lambda,$$

where $P_i^\lambda = (A + \lambda I)^{1/2} \, P_i \, (A + \lambda I)^{1/2}$

Stability:

Assume that $X \in S_A$ and there exists exactly one $\lambda \in \mathbb{R}$ such that $X \in S^\lambda = P_1^\lambda \cup P_2^\lambda$.

Let $A = \mathrm{diag}(a_1, a_2, a_3, a_4)$, and $a_1 < a_2 < a_3 < a_4$,

Proposition. If $\lambda < a_1$ or $\lambda > a_4$, then the closed trajectory through X is stable.

Proof. If $\lambda < a_1$, then $\{\cdot, \cdot\}$ is isomorphic to $so(4)$-bracket

It is easy to see that λ cannot belong to (a_1, a_2), (a_3, a_4)

If $\lambda \in (a_2, a_3)$, then $\{\cdot, \cdot\}$ is isomorphic to $so(2, 2)$-bracket. Both stable and unstable trajectories are possible.

Non-degenerate singularities

Let $\mathcal{F}$ be a complete family of commuting functions and $\mathcal{L}$ the corresponding Lagrangian fibration.

Let $x \in M^{2n}$ be an equilibrium point in the sense that $df(x) = 0$ for every $f \in \mathcal{F}$. Consider $A_f = \omega^{-1} d^2 f(x)$ as a linear operator. It is easy to see that A_f, $f \in \mathcal{F}$, generate a commutative subalgebra K in $sp(T_x M, \omega)$.

Definition 1. $x \in M$ is *non-degenerate*, if K is a Cartan subalgebra.

Definition 1′. $x \in M$ is *non-degenerate*, if K is n-dimensional and there is $f \in \mathcal{F}$ such that the "characteristic" polynomial

$$P(t) = \det(\sum d^2 f(x) - t\omega)$$

has no multiple roots.

Local classification of non-degenerate singularities

Three simplest (quadratic) singularities:

1) $f = p^2 + q^2$, $\omega = dp \wedge dq$, dim $= 2$ — elliptic case;
2) $f = pq$, $\omega = dp \wedge dq$, dim $= 2$ — hyperbolic case;
3) $f_1 = p_1 q_1 + p_2 q_2$, $f_2 = p_1 q_2 - p_2 q_1$,
$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ — focus-focus case.

Theorem (Eliasson, 1990) Any non-degenerate singularity is locally symplectomorphic to the direct product of singularities of these three types.

In particular, the tangent space $T_x M$ is naturally decomposed into direct sum of subspaces of dimension 2 and 4, each of which is invariant under any linearized vector field X_f, $f \in \mathcal{F}$.

Local analysis of equilibrium points for Euler-Manakov top

Let $X \in so(n)$ be an equilibrium point and regular. Consider the following "characteristic" equation:

$$\mathrm{rank}(\{\cdot,\cdot\}_{A-\lambda I}(X)) < \dim so(n) - \mathrm{ind}\, so(n)$$

Observation: Typically this equation has $\frac{1}{2}\dim O(X)$ distinct roots $\lambda_i \in \mathbb{C}$ so that the matrix of the bracket $\{\cdot,\cdot\}_{A+\lambda I}$ at the equilibrium point X can be presented in the block-diagonal form $\Phi_{A+\lambda I} = \Phi_0 \oplus \ldots \oplus \Phi_{\lambda_k} \oplus \ldots \oplus \Phi_{a_k \pm i b_k} \oplus \ldots$, where Φ_0 is a zero block and

$$\Phi_{\lambda_k} = \begin{pmatrix} 0 & \lambda_k - \lambda \\ \lambda - \lambda_k & \end{pmatrix}, \quad \Phi_{a_k \pm i b_k} = \begin{pmatrix} & & a_k - \lambda & b_k \\ & & -b_k & a_k - \lambda \\ \lambda - a_k & b_k & & \\ -b_k & \lambda - a_k & & \end{pmatrix}$$

Thus, there is a natural *"Jordan-Kronecker" decomposition* of the cotangent space $T_X^* so(n)$ into 2 and 4 dimensional subspaces (+ common kernel).

On the other hand, there is an *"Eliasson" decomposition* of the tangent space $T_X so(n)$ into 2 and 4 dimensional invariant subspaces for linearized (bi-hamiltonian) vector fields (+ common zero-space).

Proposition These decompositions are dual to each other.

This observation immediately simplifies everything!

Open questions

Q1. Are the "Eliasson" and "Jordan–Kronecker" decompositions always dual to each other?

Q2. What is a "generic" singular point for a family of compatible Poisson brackets?

Q3. What is a "normal form" of a family of compatible Poisson brackets at a "generic" singular point?

Q4. Is there any general procedure to verify non-degeneracy of singular points related to a family of compatible Poisson brackets?