Integrable systems on $so(n)$ and geodesically equivalent metrics

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Consider a finite-dimensional Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$.

Definition

The Poisson-Lie bracket on $\mathfrak{g}^*$ is defined by:

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle.$$

Thus, each function $H : \mathfrak{g}^* \to \mathbb{R}$ generates a Hamiltonian vector field on $\mathfrak{g}^*$ which has a natural interpretation in terms of the coadjoint representation:

$$X_H(x) = \mathrm{ad}^*_{dH(x)} x.$$

Complete Liouville integrability means that the corresponding Hamiltonian system $\dot{x} = X_H(x)$ (Euler equation) admits sufficiently many independent commuting first integrals $f_1, \dots, f_k$. The number k must be equal to $\frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g})$, where $\mathrm{ind}\, \mathfrak{g}$ is a corank of the Poisson-Lie bracket at a generic point $x \in \mathfrak{g}^*$.

Semisimple case

If $\mathfrak{g}$ is semisimple then it admits an invariant form (Killing form) which allows us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and ad with ad^*. The Euler equation on $\mathfrak{g}$ obtains the Lax form

$$\dot{x} = [dH(x), x].$$

Important particular case: quadratic Hamiltonians $H(x) = \frac{1}{2}\langle R(x), x\rangle$ where $R : \mathfrak{g} \to \mathfrak{g}$ is a symmetric operator. The Euler equation becomes

$$\dot{x} = [R(x), x]. \tag{1}$$

Problem: Describe/classify operators $R : \mathfrak{g} \to \mathfrak{g}$ for which (1) is completely integrable.

Manakov–Mischenko–Fomenko construction for $so(n)$

Here $\mathfrak{g} = so(n)$ is the Lie algebra of skew symmetric matrices.
Assume that $R : so(n) \to so(n)$ satisfies the following identity

$$[R(x), a] = [x, b], \quad x \in so(n), \tag{2}$$

for symmetric matrices $a \neq 0$ and b. Then the following statement holds

Theorem (Manakov, Mischenko, Fomenko)

Let $R : so(n) \to so(n)$ be symmetric and satisfy (2). Then

- *the system (1) admits the following Lax representation with a parameter:*

$$\frac{d}{dt}(x + \lambda a) = [R(x) + \lambda b, x + \lambda a];$$

- *the functions $\mathrm{Tr}\,(x + \lambda a)^k$ are first integrals of (1) for any $\lambda \in \mathbb{R}$ and, moreover, these integrals commute;*
- *if a is regular, then (1) is completely integrable.*

Geodesically equivalent (pseudo)-Riemannian metrics

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Instead of $\bar{g}$, it is convenient to introduce a linear operator ((1,1)-tensor):

$$L = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

L is (pseudo) self-adjoint w.r.t. both g and $\bar{g}$. Notice: $\bar{g} = \frac{1}{\det L} g L^{-1}$.

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Theorem (classical result)

g and $\bar{g}$ are geodesically equivalent if and only if L satisfies the following equation:

$$\nabla_u L = \frac{1}{2}\left(u \otimes d\,\mathrm{tr} L + (u \otimes d\,\mathrm{tr} L)^*\right)$$

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$$2L_{ij,k} = (trL)_{,i}\, g_{jk} + (trL)_{,j}\, g_{ik}.$$

Compatibility conditions

For the equation $\nabla_u L = F(u, L)$, we compute:

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In our case: $R(u \wedge v)\,L - L\,R(u \wedge v) = (u \wedge v) \cdot M + ((u \wedge v) \cdot M)^*$, where $M_{ij} = \nabla_i \nabla_j \mathrm{Tr}\, L$.

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Using g, we may think of $u \wedge v$ as a skew-symmetric operator and of M as a symmetric operator. Then taking into account that

$$((u \wedge v) \cdot M)^* = M^* \cdot (u \wedge v)^* = -M \cdot (u \wedge v),$$

we have:

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Theorem (Matveev, AB)

If g admits a non-trivial geodesically equivalent partner $\bar{g}$, then the Riemann curvature tensor of g is a Manakov–Mischenko–Fomenko operator on $so(g)$.

Thus, we have

$$[R(X), L] = [X, M].$$

where R is the curvature tensor, L is the operator which "connect" g and $\bar{g}$, and M is the Hessian of $\mathrm{Tr}\, L$, and X is an arbitrary skew-symmetric operator.

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Corollary

If the curvature tensor of a given metric is not a MMF operator, then g admits no geodesically equivalent $\bar{g}$.

Theorem (Kiosak, Matveev, AB)

Let $\dim \geq 3$. *Assume that* g *admits two "independent" metrics* g_1 *and* g_2 *geodesically equivalent to it. Let* g *and* g_1 *be strictly non-proportional. Then* g, g_1 *and* g_2 *are all of constant curvature.*

Fubini theorem

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Algebraic reformulation: Let

$$[R(X), L_1] = [X, M_1] \quad \text{and} \quad [R(X), L_2] = [X, M_2],$$

where L_1, L_2 and Id are linearly independent. If L_1 is regular, then $R : so(n) \to so(n)$ is a scalar operator, i.e., $R(X) = k \cdot X$.

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Lemma

If $[R(X), L_1] = [X, M_1]$ and $[R(X), L_2] = [X, M_2]$, then L_1 is proportional either to M_1, or to L_2.

Lemma

If L_1 is regular and $M_1 = k \cdot L_1$, then $R(X) = k \cdot X$.

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Let Y and Z be arbitrary symmetric matrices, We substitute $X = [L_2, Z]$ into $[R(X), L_1] = [X, M_1]$ and take "inner product" with Z:

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Similarly, $[[L_1, Y], M_2] = [[M_1, Y], L_2]$. Notice that R has disappeared!

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where $T = M_2 L_1 - L_2 M_1 = L_1 M_2 - M_2 L_1$.

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Taking "trace", we get:

$$(n+2)T + \mathrm{Tr}\, T \cdot Id = M_2 L_1 + L_1 M_2 - L_2 M_1 - M_1 L_2 = 2T$$

Thus, $T = 0$ and we come to a very simple identity:

$$M_2 Y L_1 + L_1 Y M_2 = M_1 Y L_2 + L_2 Y M_1$$

which holds for any symmetric matrix Y. We want to show that

- either L_1 is proportional to M_1 and M_2 is proportional to L_1,
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Lemma

If L_1 is regular and $M_1 = k \cdot L_1$, then $R(X) = k \cdot X$.

Proof. $[R(X), L_1] = [X, M_1] \quad \Rightarrow \quad [R(X) - k \cdot X, L_1] = 0.$

If L_1 is regular, then it is well known that its centralizer is generated by its powers $(L_1)^k$. In particular, the centralizer consists of symmetric operators only.

Since $R(X) - k \cdot X$ is skew-symmetric, we obtain

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