Integrable geodesic flow with positive topological entropy

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1 Introduction and main results

The main result of this paper is the following theorem.

Theorem 1 *There is a real-analytic Riemannian manifold M_A diffeomorphic to the quotient of $T^2 \times \mathbf{R}^1$ with respect to the free $\mathbf{Z}$-action generated by the map*

$$(X, z) \to (AX, z+1),$$

where $X = (x, y) \in T^2 = \mathbf{R}^2/\mathbf{Z}^2$, $z \in \mathbf{R}$, and A is the Anosov automorphism of the 2-torus T^2 defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \tag{1}$$

such that

i) the geodesic flow on M_A is (Liouville) integrable by C^∞ first integrals;

ii) the geodesic flow on M_A is not (Liouville) integrable by real-analytic first integrals;

iii) the topological entropy of the geodesic flow F_t is positive;

iv) the fundamental group $\pi_1(M_A)$ of the manifold M_A has an exponential growth;

v) the unit covector bundle SM_A contains a submanifold N such that N is diffeomorphic to the 2-torus T^2 and the restriction of F_1 onto N is the Anosov automorphism given by matrix (1).

To explain the statement in detail we recall main definitions and results on topological obstructions to integrability of geodesic flows.

Let g_{jk} be a Riemannian metric on an n-dimensional manifold M^n. It defines the geodesic flow on the tangent bundle TM^n which is a Lagrangian system with the Lagrange function

$$L(x, \dot{x}) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k.$$

The Legendre transform $TM^n \to T^*M^n$

$$\dot{x} \in T_x M^n \to p \in T^*_x M^n : \; p_j = g_{jk} \dot{x}^k$$

maps this Lagrangian system into a Hamiltonian system on T^*M^n with a symplectic form

$$\omega = \sum_{j=1}^n dx^j \wedge dp_j$$

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and the Hamilton function

$$H(x,p) = \frac{1}{2} g^{jk}(x) p_j p_k.$$

This Hamiltonian system is also called the geodesic flow of the metric.

The symplectic form defines the Poisson brackets on the space of smooth functions on T^*M^n by the formula

$$\{f, g\} = h^{jk} \frac{\partial f}{\partial y^j} \frac{\partial g}{\partial y^k},$$

where $\omega = h_{jk} dy^j \wedge dy^k$ locally.

It is said that a Hamiltonian system on a $2n$-dimensional symplectic manifold is (Liouville) integrable if there are n first integrals $I_1, \dots, I_n$ of this system such that

1) these integrals are in involution: $\{I_j, I_k\} = 0$ for any j, k, $1 \leq j, k \leq n$;

2) these integrals are functionally independent almost everywhere, i.e., on a dense open subset.

Since restrictions of the geodesic flow onto different non-zero level surfaces of its Hamilton function $H = I_n$ are smoothly trajectory equivalent, we may replace this notion of integrability by the weaker condition that there are $n-1$ additional first integrals $I_1, \dots, I_{(n-1)}$ which are in involution and functionally independent almost everywhere on the unit covector bundle $SM^n = \{H(x,p) = 1\} \subset T^*M^n$ [8].

If M^n is real-analytic together with the metric and all first integrals $I_1, \dots, I_n$, then it is said that the geodesic flow is analytically integrable.

Kozlov established the first topological obstruction to analytic integrability of geodesic flows proving that the geodesic flow of a real-analytic metric on a two-dimensional closed oriented manifold M^2 of genus $g > 1$ does not admit an additional analytic first integral [4] (see also [5] for general setup of nonintegrability problem).

For higher-dimensional manifolds, obstructions to integrability were found in [8, 9] where it was proved that analytic integrability of the geodesic flow on a manifold M^n implies that

1) the fundamental group $\pi_1(M^n)$ of M^n is almost commutative, i.e., contains a commutative subgroup of finite index;

2) if the first Betti number $b_1(M^n)$ equals k, then the real cohomology ring $H^*(M^n; \mathbf{R})$ contains a subring isomorphic to the real cohomology ring of a k-dimensional torus and, in particular, $b_1(M^n) \leq \dim M^n = n$.

In these results the analyticity condition may be replaced by stronger geometric condition called geometric simplicity and reflecting some tameness properties of the singular set where the first integrals are functionally dependent. For instance, one may only assume that SM^n is a disjoint union of a closed invariant set Γ which is nowhere dense and of finitely many open toroidal domains foliated by invariant tori.

Later Paternain proposed another approach to finding topological obstructions to integrability based on a vanishing of the topological entropy of the geodesic flow on SM^n. If this quantity vanishes, then $\pi_1(M^n)$ has a subexponential growth [2] and, if in addition M^n is a C^∞ simply-connected manifold, then Y is rationally-elliptic (this follows from results of Gromov and Yomdin) [6, 7]. Integrability implies vanishing of the topological entropy under some additional conditions which were established in [6, 7, 10] and restrict not only the singular set but also the behaviour of the flow on this set.

Recently Butler has found new examples of C^∞ integrable geodesic flows of homogeneous metrics on nilmanifolds [1]. The simplest of them is a 3-manifold M_B obtained from a product $T^2 \times [0,1]$ by identifying the components of the boundary by a homeomorphism

$$(X, 0) \to (BX, 1),$$

where

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $X \in T^2$. The fundamental group of the resulting manifold M_B is not almost commutative, $b_1(M_B) = 2$, and $H^*(M_B; \mathbf{R})$ does not contain a subring isomorphic to $H^*(T^2; \mathbf{R})$. However, $b_1(M_B) < \dim M_B$. This shows that some of the results of [8, 9] are not generalized for the C^∞ case. Note that the topological entropy vanishes for Butler's examples.

The present paper is based on an observation that Butler's construction is generalized for constructing C^∞ integrable geodesic flows on all T^n-bundles over S^1. In the case when the gluing automorphism $C : T^n \to T^n$ is hyperbolic we obtain remarkable Hamiltonian systems on a cotangent bundle to M_C: they are C^∞ integrable but have positive topological entropy. This, in particular, shows that treating positivity of topological entropy as a criterion for chaos which is used sometimes is not correct.

We confine only to one example of such a flow which we study in detail.

2 The metric on M_A and its geodesic flow

Let

$$A : \mathbf{Z}^2 \to \mathbf{Z}^2$$

be an automorphism determined by matrix (1). It determines the following action on T^2:

$$(x, y) \bmod \mathbf{Z}^2 \to (2x + y, x + y) \bmod \mathbf{Z}^2.$$

We construct M_A as follows. Take a product $T^2 \times [0,1]$ and identify the components of its boundary using the automorphism A:

$$(X, 0) \sim (AX, 1),$$

where $X = (x, y) \in T^2$. We denote the resulted manifold by M_A. Near every point $p \in M_A$ we have local coordinates x, y and t, where z is a linear coordinate on $S^1 = \mathbf{R}/\mathbf{Z}$.

Take the following metric on M_A:

$$ds^2 = dz^2 + g_{11}(z)dx^2 + 2g_{12}(z)dxdy + g_{22}(z)dy^2$$

where

$$G(t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21}(= g_{12}) & g_{22} \end{pmatrix} = \exp(-zG_0^\top)\exp(-zG_0) \qquad (2)$$

and $\exp G_0 = A$. We set

$$g_{33} = 1, g_{13} = g_{23} = 1.$$

Indeed, this formula defines a metric on an infinite cylinder $\mathcal{C} = T^2 \times \mathbf{R}$ which is invariant with respect to the $\mathbf{Z}$-action generated by

$$(x, y, z) \to (2x + y, x + y, z + 1), \tag{3}$$

and, therefore, it descends to a metric on the quotient space $M_A = \mathcal{C}/\mathbf{Z}$.

Proposition 1 *The geodesic flow of metric (2) on the infinite cylinder $\mathcal{C}$ admits three first integrals which are functionally independent almost everywhere.*

Proof of Proposition. The Hamiltonian function

$$F_3 = H = \frac{1}{2}\left(p_z^2 + g^{11}(z)p_x^2 + 2g^{12}(z)p_x p_y + g^{22}(z)p_y^2\right)$$

of this flow is, by the construction, a first integral. Since H does not depend on x and y the quasimomenta $F_1 = p_x$ and $F_2 = p_y$ are also first integrals. It is clear that the set of first integrals I_1, I_2, and I_3 is functionally independent almost everywhere. This proves the proposition.

Since action (3) preserves the symplectic form ω, it induces the following action on $T^*\mathcal{C}$:

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}, \quad p_z \to p_z.$$

This descends to a linear action on T^*M_A which preserves fibers and takes the form

$$\begin{gathered} \left(p_x - \frac{1+\sqrt{5}}{2}p_y\right) \to \lambda \left(p_x - \frac{1+\sqrt{5}}{2}p_y\right), \\ \left(p_x - \frac{1-\sqrt{5}}{2}p_y\right) \to \lambda^{-1} \left(p_x - \frac{1-\sqrt{5}}{2}p_y\right), \\ p_z \to p_z, \quad \lambda = \frac{3+\sqrt{5}}{2}. \end{gathered} \tag{4}$$

It is evident that the indefinite quadratic form

$$I_1 = \left(p_x - \frac{1+\sqrt{5}}{2}p_y\right)\left(p_x - \frac{1-\sqrt{5}}{2}p_y\right) = p_x^2 - p_x p_y - p_y^2$$

and the positively definite quadratic form

$$I_3 = H = \frac{1}{2}\left(p_z^2 + g^{11}(z)p_x^2 + 2g^{12}(z)p_x p_y + g^{22}(z)p_y^2\right)$$

are invariants of this action. To construct the third invariant we notice that

$$\frac{\log\left|p_x - \frac{1+\sqrt{5}}{2}p_y\right|}{\log \lambda}$$

is not invariant but the action adds 1 to this quantity when it is correctly defined. Therefore, the following function

$$I_2 = f(I_1) \cdot \sin\left(\frac{\log\left|p_x - \frac{1+\sqrt{5}}{2}p_y\right|}{\log \lambda}\right),$$

where

$$f(u) = \exp\left(-\frac{1}{u^2}\right),$$

is everywhere defined and invariant with respect to action (4).

Proposition 2 *The functions I_1, I_2, and I_3 are C^∞ first integrals of the geodesic flow on M_A which are functionally independent almost everywhere. Therefore, the geodesic flow on M_A is (Liouville) integrable by C^∞ functions.*

Proof of Proposition. The functions I_1, I_2 , and I_3 on $T^*\mathcal{C}$ are invariants of action (4) and, therefore, descend to functions on T^*M_A. We may consider I_1 and I_2 as replacing F_1 and F_2: they are pairwise involutive and independent on spatial variables x, y, z. Moreover they do not depend on p_z and, therefore, they are in involution with $I_3 = H$ which, in particular, means that they are first integrals of the geodesic flow. It remains to notice that, by their construction, they are C^∞. This finishes the proof of Proposition.

Proposition 3 *Let N be a subset of the unit covector bundle SM_A formed by the points with*

$$z = 0, \quad p_x = p_y = 0, \quad p_z = 1.$$

Then it is diffeomorphic to T^2 and the translation

$$F_t : T^*M_A \to T^*M_A$$

along the trajectories of the geodesic flow for $t = 1$ maps N into itself and this map is the Anosov automorphism given by matrix (1).

Proof of Proposition. The geodesic flow on M_A is covered by the geodesic flow on $\mathcal{C}$ for which p_x and p_y are first integrals. Therefore, on $\mathcal{C}$ the translation of the preimage of N under projection is as follows:

$$F_t(x, y, z, p_x, p_y, p_z) = F_t(x, y, 0, 0, 0, 1) = (x, y, t, 0, 0, 1).$$

Recalling the construction of M_A proves the proposition.

Note that Propositions 2 and 3 prove statements i) and v) of Theorem 1, respectively.

3 The fundamental group $\pi_1(M_A)$ and the topological entropy of the geodesic flow on M_A

The manifold M_A is covered by $\mathbf{R}^3$ on which acts $\pi_1(M_A)$. This group is generated by

$$a : (x, y, z) \to (x + 1, y, z), \quad b : (x, y, z) \to (x, y + 1, z),$$

$$c : (x, y, z) \to (2x + y, x + y, z + 1).$$

The relations between these generators are

$$[a, b] = 1, \quad [c, a] = ab, \quad [c, b] = a.$$

Proposition 4 (see, for instance, [3]) *$\pi_1(M_A)$ has an exponential growth.*

This follows from the hyperbolicity of A or may be proved directly: the words $ca^{\varepsilon_1}ca^{\varepsilon_2}\dots ca^{\varepsilon_k}$ are different for $\varepsilon_j = 0, 1$ and, therefore, $\gamma(2k) \geq 2^k$, where γ is the growth function of $\pi_1(M_A)$ with respect to generators a, b, and c.

Corollary 1 *The geodesic flow on M_A is not (Liouville) integrable by real-analytic first integrals.*

It follows from the results of [8] (also exposed in Section 1) that if this flow is analytically integrable, then $\pi_1(M_A)$ is almost commutative and, therefore, has a polynomial growth. This contradiction establishes the corollary.

Corollary 2 *The topological entropy of the geodesic flow on M_A is positive.*

Indeed, it was proved by Dinaburg, that if the fundamental group of a manifold M^n has an exponential growth, then the topological entropy of the geodesic flow of any Riemannian metric on M^n is positive [2].

The latter corollary also follows from Proposition 3: it is known that the topological entropy equals the supremum of the measure entropies taken over all ergodic invariant Borel measures. Hence, we may take a singular measure concentrated on $N \subset SM_A$ which has the form

$$d\mu = dx \wedge dy.$$

It is well known that the measure entropy of the Anosov automorphism $A : N \to N$ is positive (this follows, for instance, from nonvanishing of the Lyapunov exponents for any point of N).

Note that Proposition 4 and Corollaries 1 and 2 establish statements iv), ii), and iii) of Theorem 1, respectively.

The authors thank L. Butler for sending them his preprint [1] and I. K. Babenko for helpful discussions.

The authors were partially supported by Russian Foundation for Basic Researches (grants 96-15-96868 and 98-01-00240 (A. V. B.), and 96-15-96877 and 98-01-00749 (I.A.T.)) and INTAS (grant 96-0070 (I.A.T.)).

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