Non-commutative Integrability, Moment Map and Geodesic Flows

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Abstract

The purpose of this paper is to discuss the relationship between commutative and non-commutative integrability of Hamiltonian systems and to construct new examples of integrable geodesic flows on Riemannian manifolds. In particular, we prove that the geodesic flow of the bi-invariant metric on any bi-quotient of a compact Lie group is integrable in the non-commutative sense by means of polynomial integrals, and therefore, in the classical commutative sense by means of C^∞–smooth integrals.

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0 Introduction

The purpose of this paper is to establish complete integrability of a wide class of Hamiltonian systems connected with Hamiltonian actions of Lie groups, with a special attention to the integrability of the geodesic flows of Riemannian metrics.

Let (M,ω) be $2n$–dimensional connected symplectic manifold. By sgrad h we shall denote the Hamiltonian vector field of a function h, and by $\{\cdot,\cdot\}$ the canonical Poisson brackets on M. Consider Hamiltonian equations:

$$\dot{x} = \operatorname{sgrad} h(x). \tag{0.1}$$

A function f is an integral of the Hamiltonian system (constant along trajectories of (0.1)) if and only if it commutes with h: $\{h,f\}=0$.

One of the central problems in Hamiltonian dynamics is whether the equations (0.1) are completely integrable or not. The usual definition of complete integrability is as follows (see for instance [20]):

Definition 0.1 Hamiltonian equations (0.1) are called *completely integrable* if there are n Poisson-commuting smooth integrals $f_1, \dots, f_n$ whose differentials are independent in an open dense subset of M.

If system (0.1) is completely integrable then by Liouville's theorem the general solutions of the system (0.1) can be (locally) solved in quadratures. Moreover, compact connected components of regular invariant submanifolds $\{f_1 = c_1, \dots, f_n = c_n\}$ are diffeomorphic to n–dimensional Lagrangian tori with linear dynamics (see [1, 2, 18, 20, 33]).

The algebra of integrals $\mathcal{F} = \{f_1, \dots, f_n\}$ is called a *complete involutive (or commutative) algebra* of functions on M.

Let M be the cotangent bundle of a Riemannian manifold (Q, g) with the natural symplectic structure. Taking $h(\xi, x) = \frac{1}{2} g_x^{-1}(\xi, \xi)$, $\xi \in T_x^* Q$ as a Hamiltonian in (0.1), we obtain the equations of the geodesic flow. It is very rare for this flow to be completely integrable. Almost all known manifolds with integrable geodesic flows are diffeomorphic to some symmetric spaces. The point is that there are very serious topological obstructions to the integrability. As an example, we mention Taimanov's theorem [30, 31] saying that in the real analytic case the fundamental group a manifold Q admitting integrable geodesic flows must be almost commutative (Q has a finite-sheeted covering $p : \tilde{Q} \to Q$, where the group $\pi_1(\tilde{Q})$ is commutative).

The classical examples of Riemannian manifolds with integrable geodesic flows are surfaces of revolution (Clairaut), flat tori, n–dimensional ellipsoids (Jacobi) and the Lie group $SO(3)$ with a left invariant metric (Euler). Mishchenko and Fomenko proved integrability of certain left-invariant metrics on compact Lie groups [22, 33]. This result was generalized to all compact symmetric spaces by Mishchenko [21], Thimm [32], Mikityuk [24] and Brailov [10]. Examples of homogeneous, but not symmetric, spaces with integrable geodesic flows can be found in [6, 7, 11, 16, 25, 28, 32]. Recently the authors [8] have proved the noncommutative integrability of geodesic flows on all homogeneous spaces $\mathfrak{G}/\mathfrak{H}$, where $\mathfrak{G}$ is a compact connected Lie group.

Natural generalizations of homogeneous spaces are the so-called bi-quotients of Lie groups. These manifolds were introduced by Gromoll and Meyer [15] in order to construct metrics with nonnegative sectional curvature on the exotic 7–sphere.

The examples of bi-quotients with integrable geodesic flows (which include, in particular, Eschenburg's manifolds [14] and the exotic 7–sphere of Gromoll and Meyer [15]) have been found and studied by Paternain and Spatzier [28] and Bazaikin [3].

Now we shall present the main results of the paper:

- We prove that non-commutative integrability always implies complete commutative integrability in the sense of definition 0.1 by means of C^∞–

smooth integrals (section 1). This result proves the conjecture of Mishchenko and Fomenko [33] about the relationship between these two kinds of integrability in the C^∞–case.

- Consider the Hamiltonian action of a Lie group $\mathfrak{G}$ on a symplectic manifold M with a moment map $\Phi : M \to \mathfrak{g}^*$. Under some assumption (that holds if the action is proper) we prove that from $\mathfrak{G}$ invariant functions and so-called collective functions (functions of the form: $f_h = h \circ \Phi$) we can always construct completely integrable systems on M. In that way we generalize the results of Guillemin and Sternberg [16, 17, 18] obtained for multiplicity-free actions (section 2).

- As a simple application of the above results, we prove the complete integrability of geodesic flows on manifolds all of whose geodesics are closed (section 3) and give a new prove of the theorem that geodesic flows on homogeneous spaces $\mathfrak{G}/\mathfrak{H}$ of compact Lie groups $\mathfrak{G}$ are completely integrable (section 4).

- Finally, we prove integrability of geodesic flows of bi-invariant metrics on all bi-quotients $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$ of compact Lie groups $\mathfrak{G}$ (section 5).

The notation of the paper is usual. The algebra of smooth functions on a manifold M is denoted by $C^\infty(M)$. If something holds *for general* $x \in M$, this means that the property holds for all x from an open dense subset of M. By $\mathfrak{G}, \mathfrak{H}, \mathfrak{K}, \dots$ we denote Lie groups and by $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \dots$ the corresponding Lie algebras. The orbit of the co-adjoint action through $\mu \in \mathfrak{g}^*$ is denoted by $\mathcal{O}(\mu)$. For a $\mathfrak{G}$ action on M, the orbit through $x \in M$ is denoted by $\mathfrak{G} \cdot x$ and the isotrop subgroup of x by $\mathfrak{G}_x$. The annihilator of the linear subspace $V \subset W$ in the dual space W^* is denoted by ann $(V) \subset W^*$.

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1 Non-commutative integrability

We shall briefly recall the concept of non-commutative integrability introduced by Mishchenko and Fomenko in [23].

If f_1 and f_2 are integrals of (0.1), then so are an arbitrary smooth function $F(f_1, f_2)$ and the Poisson bracket $\{f_1, f_2\}$. Therefore without loss of generality we can assume that integrals of (0.1) form an algebra $\mathcal{F}$ with respect to the Poisson bracket. For simplicity we shall assume that $\mathcal{F}$ is functionally generated by functions $f_1, \dots, f_l$ so that:

$$\{f_i, f_j\} = a_{ij}(f_1, \dots, f_l). \tag{1.1}$$

Suppose that

$$\begin{aligned}
&\dim F_x = \dim \operatorname{span}\{df_i(x), i = 1, \dots, l\} = l, \quad x \in U,\\
&\dim \ker\{\cdot,\cdot\}|_{F_x} = r, \quad x \in U,
\end{aligned}$$

holds for an open dense set $U \subset M$.

Let $\phi : M \to \mathbb{R}^l$ be the moment mapping:

$$\phi(x) = (f_1(x), \dots, f_l(x)) \tag{1.2}$$

and let $\Sigma = \phi(M \setminus U)$.

Then we have the following non-commutative integration theorem (see also Mishchenko and Fomenko [23], Nekhoroshev [9] and Brailov [26]):

Theorem 1.1 *Suppose that:*

$$\dim F_x + \dim \ker\{\cdot,\cdot\}|_{F_x} = \dim M, \tag{1.3}$$

for $x \in U$. Let $c \in \phi(M) \setminus \Sigma$ be a regular value of the moment map. Then:

(i) $M_c = \phi^{-1}(c)$ is an isotropic submanifold of M and the equations (0.1) on M_c can be (locally) solved by quadratures;

(ii) Compact connected components T_c^r of M_c are diffeomorphic to r–dimensional tori. In the neighborhood of T_c^r there are generalized action-angle variables $y, x, I, \varphi \bmod 2\pi$ such that the symplectic form becomes:

$$\omega = \sum_{i=1}^{r} dI_i \wedge d\varphi_i + \sum_{i=1}^{q} dy_i \wedge dx_i,$$

and the Hamiltonian function h depends only on $I_1, \dots, I_r$. The invariant tori are given as the level sets of integrals I_i, y_j, x_k. The equations (0.1) on invariant tori take the linear form:

$$\dot\varphi_1 = \omega_1(I) = \frac{\partial h}{\partial I_1}, \dots, \dot\varphi_r = \omega_r(I) = \frac{\partial h}{\partial I_r}.$$

Let us just point out some steps in the proof of the theorem.

Under the hypotheses of theorem 1.1, there exist r linearly independent commuting vector fields $X_1 = \text{sgrad}\, h, X_2, \dots, X_r$ on M_c. They can be obtained as linear combinations of the Hamiltonian vector fields $\text{sgrad}\, f_1, \dots, \text{sgrad}\, f_l$ from the conditions $\omega(X_i, \text{sgrad}\, f_j) = 0$, $i = 1, \dots, r$, $j = 1, \dots, l$. Since commutative algebras are solvable we can apply the theorem of S.Lie to integrate the system $\dot{x} = \text{sgrad}\, h(x)$.

The Lie theorem says that if in some domain $V \subset \mathbb{R}^n\{x\}$, we have n linearly independent vector fields $X_1, \dots, X_n$ that generate a solvable Lie algebra under commutation, and if $[X_1, X_i] = \lambda_i X_i$, then the differential equation $\dot{x} = X_1(x)$ can be integrated by quadratures in V (see [2, 34]). In our case V is an open set in M_c, $n = r$.

On a compact connected component T_c^r of M_c, the vector fields $X_1, \dots, X_r$ are complete. Therefore T_c^r is diffeomorphic to an r–dimensional torus and the motion on the torus is quasi-periodic (the proof of this fact is just the same as in the usual Liouville theorem [1]). The existence of generalized action-angle variables follows from the Nekhoroshev theorem [26].

If the algebra of integrals $\mathcal{F} = \{f_1, \dots, f_l\}$ is commutative, i.e. $\{f_i, f_j\} = 0$ and $n = l = r$ then from theorem 1.1 we obtain the Liouville theorem. But, in fact theorem 1.1 is essentially stronger then Liouville's theorem: under the assumptions of theorem 1.1, the Hamiltonian system (0.1) is integrable in the usual, commutative sense.

In order to prove this statement, we shall modify a remark by Brailov (see [34]) that on every symplectic manifold there exists a complete involutive set of functions. Brailov's idea was to fill up the manifold with disjoint Darboux symplectic balls. On every symplectic ball one can construct complete involutive set of functions that can be then "glued" in order to obtain involutive functions globally defined.

Theorem 1.2 *Under the assumptions of theorem 1.1, the Hamiltonian system (0.1) is integrable in the usual, commutative sense, i.e., it admits n Poisson-commuting C^∞–smooth integrals $g_1, \dots, g_n$, independent on an open dense subset of M.*

Proof. On the image of M under the moment mapping $\phi(M) \subset \mathbb{R}^l\{y_1, \dots, y_l\}$ we can introduce a "Poisson structure" $\{\cdot, \cdot\}_\mathcal{F}$ by the formulas:

$$\{y_i, y_j\}_\mathcal{F} = a_{ij}(y_1, \dots, y_l).$$

Note that $\text{rank}\, \{\cdot, \cdot\}_\mathcal{F}(y) = l - r = 2q$ for all $y \in \phi(M) \setminus \Sigma$ and that $(\phi(M) \setminus \Sigma, \{\cdot, \cdot\}_\mathcal{F})$ is a Poisson manifold.

From the definition of the Poisson brackets $\{\cdot, \cdot\}_\mathcal{F}$ it follows that if smooth functions $F, G : \mathbb{R}^l \to \mathbb{R}$ are in involution with respect to $\{\cdot, \cdot\}_\mathcal{F}$ on $\phi(M)$, then their liftings $f = F \circ \phi$ and $g = G \circ \phi$ commute on M:

$$\{f, g\}(x) = \{F, G\}_\mathcal{F}(\phi(x)) = 0. \tag{1.4}$$

Let $z \in \phi(M) \setminus \Sigma$. By the theorem on the local structure of Poisson brackets (see [20]), there is a neighborhood $U(z) \subset \phi(M) \setminus \Sigma$ of z and l independent smooth functions $G_1, \dots, G_l : U(z) \to \mathbb{R}$, $G_i(z) = 0$ such that

$$\{G_i, G_{i+q}\}_{\mathcal{F}} = 1 = -\{G_{i+q}, G_i\}_{\mathcal{F}} \quad i = 1, \dots, q \tag{1.5}$$

and the remaining Poisson brackets vanish. Let a ball $B^\alpha(\epsilon_\alpha)$ belong to $U(z)$, where

$$B^\alpha(\epsilon_\alpha) = \{y \in U(z), G_1^2 + \dots + G_l^2 < \epsilon_\alpha\} \tag{1.6}$$

Then, starting from the n involutive functions on B^α:

$$h_1 = G_1^2 + G_{1+q}^2, \dots, h_q = G_q^2 + G_{2q}^2, h_{q+1} = G_{2q+1}^2, \dots, h_n = G_l^2 \tag{1.7}$$

we can construct a smooth set of nonnegative functions $F_1^\alpha, \dots, F_n^\alpha : \mathbb{R}^l \to \mathbb{R}$ that are independent on an open dense subset of $B^\alpha(\epsilon)$, equal to zero outside $B^\alpha(\epsilon_\alpha)$, in involution on $\phi(M)$, and satisfies inequalities $F_i^\alpha < e^{-\epsilon_\alpha}$ together with all derivatives. To this end we use the construction suggested by Brailov for Darboux symplectic balls (see [34]).

Let $g : \mathbb{R} \to \mathbb{R}$ be smooth nonnegative function, such that $g(x)$ is equal to zero for $|x| > \epsilon_\alpha$, monotonically increases on $[-\epsilon, 0]$ and monotonically decreases on $[0, \epsilon]$. Let $h(y) = g(h_1(y) + \dots + h_n(y))$. This function could be extended by zero to the whole manifold. Now, we can define F_i^α by: $F_i^\alpha = h \cdot h_i$. Obviously, $\{F_i^\alpha, F_j^\alpha\}_{\mathcal{F}} = 0$. These functions are independent inside B^α. Also, we can choose g in such a way that F_i^α satisfies the inequalities $F_i^\alpha < e^{-\epsilon_\alpha}$ together with all derivatives.

In the same way we can construct a countable family of open balls $\{B^\alpha(\epsilon_\alpha)\}$, $B^\alpha \cap B^\beta = \emptyset$, and functions $\{F_1^\alpha, \dots, F_l^\alpha\}$ with the above properties, such that $B = \cup_\alpha B^\alpha(\epsilon_\alpha)$ is an open everywhere dense set of $\phi(M) \setminus \Sigma$. Let us define the functions $F_1, \dots, F_n : \mathbb{R}^l \to \mathbb{R}$ by:

$$\begin{aligned} &F_i(y) = F_i^\alpha, \quad y \in B^\alpha \subset B \\ &F_i(y) = 0, \quad y \in \mathbb{R}^l \setminus B, \quad i = 1, \dots, n \end{aligned}$$

By (1.4), the functions $g_1 = F_1 \circ \phi, \dots, g_n = F_n \circ \phi$ will have the desired properties. q.e.d.

Theorem 1.2 says that the r–dimensional invariant tori T^r can be organized into larger, n–dimensional Lagrangian tori T^n that are level sets of a commutative algebra of integrals. Since the tori T^n are fibered into invariant tori T^r, the trajectories of (0.1) are not dense on T^n. In this sense, the system (0.1) is degenerate. Thus, establishing the fact of non-commutative integrability of the system give us more information on the behavior of its integral trajectories than we could obtain from the usual Liouville integrability. Note that, contrary to the case of non-degenerate integrable systems, the fibration by Lagrangian tori is neither intrinsic nor unique.

Let $P_0 = \phi(M) \setminus \Sigma$, $M_0 = \phi^{-1}(P_0)$. If all invariant submanifolds $M_c = \phi^{-1}(c)$, $c \in P_0$ are compact and connected, then $\phi : (M_0, \{\cdot,\cdot\}) \to (P_0, \{\cdot,\cdot\}_{\mathcal{F}})$ is a Poisson morphism and isotropic fibration. This fibration is symplectically complete, i.e., the symplectic orthogonal distribution to the tangent spaces of the fibres is a foliation (see [20]). The geometry of such fibrations as well as obstructions to the existence of global generalized action-angle variables are studied in [12, 26].

By theorems 1.1 and 1.2, we can give the following definition of completely integrable systems:

Definition 1.1 Let $\mathcal{F} \subset C^\infty(M)$ be an algebra of functions, closed under the Poisson brackets. Let F_x be the subspace of $T^*_x M$ generated by $df(x)$, $f \in \mathcal{F}$. Let $K_x \subset F_x$ be the kernel of Poisson structure restricted on F_x. Suppose that $\dim F_x = l$, $\dim K_x = r$ holds on an open dense subset $U \subset M$. We shall denote U by $\mathrm{reg}\,\mathcal{F}$ (*regular points* of $\mathcal{F}$). The numbers l and r are usually denoted by $\mathrm{ddim}\,\mathcal{F}$ and $\mathrm{dind}\,\mathcal{F}$ and are called *differential dimension* and *differential index* of $\mathcal{F}$. The algebra $\mathcal{F}$ is said to be *complete* if:

$$\mathrm{ddim}\,\mathcal{F} + \mathrm{dind}\,\mathcal{F} = \dim M.$$

Remark 1.1 Let $\mathcal{F}$ be a complete algebra, $x \in \mathrm{reg}\,\mathcal{F}$ and

$$W_x = \mathrm{span}\,\{\mathrm{sgrad}\, f(x),\ f \in \mathcal{F}\}.$$

Then the completeness condition $\dim F_x + \dim \ker\{\cdot,\cdot\}_{F_x} = \dim M$ is equivalent to the coisotropy of W_x in the symplectic linear space $T_x M$: $W_x^\omega \subset W_x$.

Remark 1.2 Suppose that independent functions $f_1, \dots, f_l$ generate a finite dimensional Lie algebra $\mathfrak{g} = \mathcal{F} = \oplus_{i=1}^l \mathbb{R} f_i$ under the Poisson brackets:

$$\{f_i, f_j\} = \sum_{k=1}^{l} c_{ij}^k f_k,$$

c_{ij}^k are constants. In that case the numbers $\mathrm{ddim}\,\mathcal{F}$ and $\mathrm{dind}\,\mathcal{F}$ coincide with the dimension and the index of the Lie algebra $\mathfrak{g}$ [23].

Definition 1.2 The Hamiltonian system (0.1) is *completely integrable* if it possesses a complete algebra $\mathcal{F}$ of integrals.

Remark 1.3 In definition 1.2 we do not require that $\mathcal{F}$ is generated by $l = \mathrm{ddim}\,\mathcal{F}$ functions. We shall briefly explain this. Let x_0 belong to $\mathrm{reg}\,\mathcal{F}$. Then there are integrals $f_1, \dots, f_l \in \mathcal{F}$ that are independent in x_0, where $l = \mathrm{ddim}\,\mathcal{F}$. Let V be the open set where the functions $f_1, \dots, f_l$ are independent. Since $f_1, \dots, f_l$ are integrals of (0.1) the trajectory of (0.1) that has initial position in V remains in V. Indeed, the phase flow of (0.1) preserves the form $df_1 \wedge \ldots \wedge df_l$. Therefore we can consider the restriction of (0.1) to V and apply theorem 1.1 to integrate it. If all connected regular invariant submanifolds are isotropic tori then by the same construction as in theorem 1.2, we can construct a commutative set of smooth first integrals that are independent on an open dense set of M. The original isotropic tori are organized in Lagrangian tori that are level sets of a commutative algebra of integrals.

Mishchenko and Fomenko stated the conjecture that non-commutative integrable systems are integrable in the usual commutative sense by means of integrals that belong to the same functional class as the original non-commutative algebra of integrals. In particular, when $\mathcal{F}$ is a finite-dimensional Lie algebra, this conjecture is proved for compact manifolds M and for all semi-simple Lie algebras (see [5, 33, 34] and references therein).

Our theorem 1.2 proves the conjecture in general without assuming that $\mathcal{F}$ is finite dimensional. However the above construction gives the proof only for C^∞-smooth integrals and does not allow us to get real analytic ones. Thus the following general conjecture remains:

Conjecture 1.1 Suppose that on a real-analytic symplectic $2n$-dimensional manifold M we have a Hamiltonian system $\dot{x} = \operatorname{sgrad} h(x)$ completely integrable by means of a non-commutative algebra $\mathcal{F}$ of integrals which are real analytic functions. Then the system possesses n commuting real analytic integrals.

2 Moment map. Collective motion

Let a connected Lie group $\mathfrak{G}$ act on $2n$–dimensional connected symplectic manifold (M, ω). Suppose the action is Hamiltonian. This means that $\mathfrak{G}$ acts on M by symplectomorphisms and there is a well-defined momentum mapping:

$$\Phi : M \to \mathfrak{g}^* \tag{2.1}$$

($\mathfrak{g}^*$ is a dual space of the Lie algebra $\mathfrak{g}$) such that one-parameter subgroups of symplectomorphisms are generated by the Hamiltonian vector fields of functions $f_\xi(x) = \Phi(x)(\xi)$, $\xi \in \mathfrak{g}$, $x \in M$ and $f_{[\xi_1,\xi_2]} = \{f_{\xi_1}, f_{\xi_2}\}$. Then Φ is equivariant with respect to the given action of $\mathfrak{G}$ on M and the co-adjoint action of $\mathfrak{G}$ on $\mathfrak{g}^*$:

$$\Phi(g \cdot x) = Ad_g^*(\Phi(x)). \tag{2.2}$$

In particular, if μ belongs to $\Phi(M)$ then the co-adjoint orbit $\mathcal{O}(\mu)$ belongs to $\Phi(M)$ as well.

Throughout the paper we shall use the following notation for two natural classes of functions on M. By $\mathcal{F}_1$ we shall denote the subalgebra of functions in $C^\infty(M)$ obtained by pulling-back the algebra $C^\infty(\mathfrak{g}^*)$ by the moment map (2.1):

$$\mathcal{F}_1 = \Phi^* C^\infty(\mathfrak{g}^*) = \{f_h = h \circ \Phi,\ h : \mathfrak{g}^* \to \mathbb{R}\} \tag{2.2}$$

and by $\mathcal{F}_2$ we shall denote the algebra of $\mathfrak{G}$–invariant functions in $C^\infty(M)$:

$$\mathcal{F}_2 = \{f : M \to \mathbb{R},\ f(g \cdot x) = f(x),\ x \in M, g \in \mathfrak{G}\}. \tag{2.3}$$

Let $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ be the Lie-Poisson bracket on $\mathfrak{g}^*$:

$$\{f(\mu), g(\mu)\}_{\mathfrak{g}^*} = \mu([df(\mu), dg(\mu)]), \quad f, g : \mathfrak{g}^* \to \mathbb{R}. \tag{2.4}$$

Then the mapping $h \mapsto f_h$ is a morphism of Poisson structures:

$$\{f_{h_1}(x), f_{h_2}(x)\} = \{h_1(\mu), h_2(\mu)\}_{\mathfrak{g}^*}, \quad \mu = \phi(x). \tag{2.5}$$

Thus, $\mathcal{F}_1$ is closed under the Poisson brackets. Since $\mathfrak{G}$ acts in a Hamiltonian way, $\mathcal{F}_2$ is closed under the Poisson brackets as well.

It follows from the Noether theorem that the moment map Φ is an integral of the Hamiltonian equations for all Hamiltonian functions h that belong to $\mathcal{F}_2$. In other words:

$$\{\mathcal{F}_1, \mathcal{F}_2\} = 0. \tag{2.6}$$

Assumption 2.1 Let a general orbit of the action have dimension m. We shall suppose that

$$\mathrm{span}\,\{df(x),\ f \in \mathcal{F}_2\} = \mathrm{ann}\,(T_x(\mathfrak{G} \cdot x)), \tag{2.7}$$

for general $x \in M$, $\dim \mathfrak{G} \cdot x = m$. Whence $\mathrm{ddim}\,\mathcal{F}_2 = 2n - m$. By $\mathrm{reg}\,\mathcal{F}_2$ denote the open dense set where (2.7) holds.

The assumption 2.1 holds for any proper group action because all orbits are separated by invariant functions. Moreover, for the action of a compact group $\mathfrak{G}$ the algebra $\mathcal{F}_2$ is generated by a finite number of functions. To be more precise, let the $\mathfrak{G}$ action have a finite number of orbit types. Then there exist functions $f_1, \dots, f_r \in \mathcal{F}_2$, such that every function $f \in \mathcal{F}_2$ is of the form: $f = F(f_1, \dots, f_r)$. This theorem was proved by Schwarz [29]. If M is compact then M has a finite number of orbit types. Furthermore, Mann proved that if M is an orientable manifold whose homology groups $H_i(M, \mathbb{Z})$ are finitely generated then the number of orbit types of any action of a compact Lie group on M is finite. A review of results concerning invariant functions of $\mathfrak{G}$ actions can be found in [27].

The following theorem, although it is a reformulation of some well known facts about the momentum mapping, is fundamental in the considerations below.

Theorem 2.1 *Suppose that assumption 2.1 holds. Then the algebra of functions $\mathcal{F}_1 + \mathcal{F}_2$ is complete:*

$$\mathrm{ddim}\,(\mathcal{F}_1 + \mathcal{F}_2) + \mathrm{dind}\,(\mathcal{F}_1 + \mathcal{F}_2) = \dim M.$$

Proof. We shall need a well known fact that $\ker d\Phi(x)$ is symplectically orthogonal to the tangent space at x to the orbit of x (see [20]):

$$\ker d\Phi(x) = (T_x(\mathfrak{G} \cdot x))^\omega. \tag{2.8}$$

Let $U = \mathrm{reg}\,(\mathcal{F}_1 + \mathcal{F}_2) = \mathrm{reg}\,\mathcal{F}_2$ be the open dense set in M, such that for $x \in U$ the moment map (2.1) has maximal rank (by (2.8) it is equivalent to the fact that the orbit $\mathfrak{G} \cdot x$ has maximal dimension) and that (2.7) holds. Let x belong to $\mathrm{reg}\,(\mathcal{F}_1 + \mathcal{F}_2)$. Consider the linear spaces:

$$W_1 = W_1(x) = \mathrm{span}\,\{\mathrm{sgrad}\, f(x),\ f \in \mathcal{F}_1\} \subset T_x M$$
$$W_2 = W_2(x) = \mathrm{span}\,\{\mathrm{sgrad}\, f(x),\ f \in \mathcal{F}_2\} \subset T_x M$$

We shall prove that the symplectic orthogonal complement of W_1 coincides with W_2:

$$W_1^\omega = W_2 \tag{2.9}$$

Then $(W_1 + W_2)^\omega \subset W_1 + W_2$ which is equivalent to the completeness of the algebra $\mathcal{F}_1 + \mathcal{F}_2$ (see remark 1.1).

It remains to prove (2.9) (note that from (2.6) we have that W_1 and W_2 are symplectic orthogonal: $\omega(W_1, W_2) = 0$). This condition is equivalent to the following condition:

$$\mathrm{ann}\,(\{df(x),\ f \in \mathcal{F}_1\}) = \ker d\Phi(x) = \mathrm{span}\,\{\mathrm{sgrad}\, f(x),\ f \in \mathcal{F}_2\}. \tag{2.10}$$

On the other side, from (2.7) we get:

$$\mathrm{span}\,\{\mathrm{sgrad}\, f(x),\ f \in \mathcal{F}_2\} = (T_x(\mathfrak{G} \cdot x))^\omega,$$

which together with (2.8) prove (2.10). The theorem is proved. q.e.d.

Remark 2.1 Note that if $h : \mathfrak{g}^* \to \mathbb{R}$ is an invariant of the co-adjoint action then $f_h = h \circ \Phi$ will be a $\mathfrak{G}$–invariant function on M. Also, we have the following property of functions in $\mathcal{F}_1$. Let $f_h = h \circ \Phi \in \mathcal{F}_1$ and let $\mu_0 = \Phi(x_0)$. If $ad^*_{dh(\mu_0)}\mu_0$ is equal to zero (or equivalently if $dh(\mu_0) \in \mathrm{ann}\, T_{\mu_0}\mathcal{O}(\mu_0)$), then by (2.4) and (2.5) we get that f_h commutes with all functions from $\mathcal{F}_2$ at x_0. By the proof of theorem 2.1, for $x_0 \in \mathrm{reg}\,(\mathcal{F}_1 + \mathcal{F}_2)$, $df_h(x_0)$ belongs to the span $\{df(x_0),\ f \in \mathcal{F}_2\}$. Thus we have:

$$\begin{aligned}
\mathrm{ddim}\,(\mathcal{F}_1 + \mathcal{F}_2) &= \dim M - \dim \mathfrak{G} \cdot x + \dim \mathcal{O}(\mu) \\
&= \dim M + \dim \mathfrak{G}_x - \dim \mathfrak{G}_\mu \\
\mathrm{dind}\,(\mathcal{F}_1 + \mathcal{F}_2) &= \dim \mathfrak{G}_\mu - \dim \mathfrak{G}_x,
\end{aligned}$$

for general $x \in M$, $\mu = \Phi(x)$ ($\mathfrak{G}_\mu$ and $\mathfrak{G}_x$ denotes the isotropy groups of $\mathfrak{G}$ action at μ and x).

The following corollary can be derived from theorem 2.1 and the above observations:

Corollary 2.1 *Suppose that assumption 2.1 holds. Let $\mathcal{A}$ be any algebra of functions on $\mathfrak{g}^*$ and $\mathcal{F}_1(\mathcal{A})$ be the pull-back of $\mathcal{A}$ by the moment map:*

$$\mathcal{F}_1(\mathcal{A}) = \Phi^*(\mathcal{A}) = \{f_h = h \circ \phi,\ h \in \mathcal{A}\}.$$

Then $\mathcal{F}_1(\mathcal{A}) + \mathcal{F}_2$ is a complete algebra on M if and only if $\mathcal{A}$ is a complete algebra on the orbit $\mathcal{O}(\mu)$ of the co-adjoint action of the group $\mathfrak{G}$ on $\mathfrak{g}^*$, for general $\mu \in \phi(M)$.

Corollary 2.1 is connected with integrability of the so-called collective motion (we follow the terminology of Guillemin and Sternberg [18]). The Hamiltonian $H : M \to \mathbb{R}$ is said to be *collective* if H is of the form $H = h \circ \Phi$, or in our notation if H belongs to $\mathcal{F}_1$.

Let a general co-adjoint orbit $\mathcal{O}(\mu) \subset \Phi(M)$ have dimension $2l$.

Theorem 2.2 *Suppose that assumption 2.1 holds. Let $h : \mathfrak{g}^* \to \mathbb{R}$ be a Hamiltonian function such that the Euler equations:*

$$\dot{\mu} = ad^*_{dh(\mu)}\mu,$$

are completely integrable on general co-adjoint orbits $\mathcal{O}(\mu) \subset \Phi(M)$ with a set of Lie-Poisson commuting integrals $f_1, \dots, f_l : \mathfrak{g}^* \to \mathbb{R}$. Then the Hamiltonian equations on M with Hamiltonian function $H = h \circ \Phi$ are completely integrable. The complete algebra of first integrals is

$$\{f_1 \circ \Phi, \dots, f_l \circ \Phi\} + \mathcal{F}_2.$$

The above theorem generalizes the results of Guillemin and Sternberg obtained for the case when the action $\mathfrak{G}$ on M is *multiplicity-free* (see [16, 17, 18]).

Remark 2.2 If $\mathfrak{G}$ is a compact group, then the connected components of regular invariant submanifolds of the integrable systems from theorems 2.1 and 2.2 are isotropic tori of dimension $\dim \mathfrak{G}_\mu - \dim \mathfrak{G}_x$ and $\frac{1}{2}(\dim \mathfrak{G} + \dim \mathfrak{G}_\mu) - \dim \mathfrak{G}_x$ respectively.

3 Manifolds all of whose geodesics are closed

Suppose that an n–dimensional Riemannian manifold (Q, ds^2) has the property that for every $x \in M$, all geodesics starting from x return back to the same point. To be more precise, let $\gamma(t)$, $|\dot{\gamma}(t)| = 1$ be a geodesic line. Then there is $T \in \mathbb{R}$ such that $\gamma(0) = \gamma(T)$, $\dot{\gamma}(0) = \dot{\gamma}(T)$. This condition implies, by the theorem of Wadsley [4, 35], that there is a common period for all geodesics. Such Riemannian manifolds are called *P–manifolds.*

It is clear that then the geodesic flow induces an S^1-action on T^*M, with the moment map given by the Hamiltonian function.

From theorem 2.1 we obtain:

Theorem 3.1 *The geodesic flow on a Riemannian P–manifold is completely integrable.*

Theorem 3.1 says that periodic trajectories are organized in n–dimensional Lagrangian tori. The frequencies of all tori are resonant. More about the geometry of manifolds all of whose geodesics are closed, together with examples of such manifolds, can be found in [4].

Remark 3.1 A stronger theorem was stated by Duran [13]. He claimed that commuting integrals could be taken in such a way that the singular set would be a polyhedron. But there seem to be some problems in his proof (lemma 3.1 in [13]).

4 Geodesic flows on homogeneous spaces

An important application of theorems 2.1 and 2.2 is the case when M is the (co)tangent bundle of a homogeneous space $\mathfrak{G}/\mathfrak{H}$, where $\mathfrak{G}$ is a compact connected Lie group. In that way we have another proof of results from [8].

Let $\mathfrak{g} = T_e\mathfrak{G}$, $\mathfrak{h} = T_e\mathfrak{H}$ be the Lie algebras of $\mathfrak{G}$ and $\mathfrak{H}$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{v}$ be the orthogonal decomposition of $\mathfrak{g}$ according to a non-degenerate $Ad_{\mathfrak{G}}$-invariant scalar product $\langle\cdot,\cdot\rangle$. We can identify $\mathfrak{v}$ with the tangent space $T_{\pi(e)}(\mathfrak{G}/\mathfrak{H})$, where $\pi : \mathfrak{G} \to M$ is the natural projection. Denote by $g\xi$ the action of $g \in \mathfrak{G}$ on the element $\xi \in \mathfrak{v} = T_{\pi(e)}M$. All functions will be analytic, polynomial in velocities.

Let ds_0^2 be the $\mathfrak{G}$ invariant metric on $\mathfrak{G}/\mathfrak{H}$ defined by: $\langle g\xi, g\eta\rangle_{\pi(g)} = \langle\xi,\eta\rangle$. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by the scalar product $\langle\cdot,\cdot\rangle$.

Consider $T(\mathfrak{G}/\mathfrak{H})$ as a symplectic manifold whose symplectic form is the pullback of the canonical symplectic form on $T^*(\mathfrak{G}/\mathfrak{H})$ by the metric ds_0^2. Then the natural $\mathfrak{G}$ action on $T(\mathfrak{G}/\mathfrak{H})$ is Hamiltonian with the moment map of the form: $\Phi(g\xi) = Ad_g\xi$, $\xi \in \mathfrak{v}$.

There are various constructions of complete algebras of involutive functions for compact Lie algebras. In order to apply theorem 2.2 we shall focus on the following construction.

Denote by $I(\mathfrak{g})$ the algebra of $Ad_{\mathfrak{G}}$ invariant polynomials on $\mathfrak{g}$. Mishchenko and Fomenko showed that the polynomials $\mathcal{A}_a = \{p_a^i\}$ obtained from invariant polynomials by shifting the argument:

$$p(\xi + \lambda a) = \sum p_a^i(\xi)\lambda^i, \quad \xi \in \mathfrak{g}, \quad p \in I(\mathfrak{g})$$

are in involution [22]. Furthermore, for general regular $a \in \mathfrak{g}$, the family $\mathcal{A}_a$ forms a complete involutive set of functions on *every* adjoint orbit in $\mathfrak{g}$. For regular orbits this is proved by Mishchenko and Fomenko [22]. For singular orbits there are several different proofs: by Mikityuk [24], Brailov [10] and Bolsinov [5].

Since a is regular, $\mathfrak{g}_a = \{\eta \in \mathfrak{g},\ [\eta, a] = 0\}$ is a Cartan subalgebra. Let b belong to $\mathfrak{g}_a$ and let $D : \mathfrak{g}_a \to \mathfrak{g}_a$ be a symmetric operator. By $\varphi_{a,b,D}$

denote the symmetric operator (called *sectional operator*) defined according to the orthogonal decomposition: $\mathfrak{g} = \mathfrak{g}_a + [a, \mathfrak{g}]$:

$$\varphi_{a,b,D}|_{\mathfrak{g}_a} = D, \quad \varphi_{a,b,D}|_{[a,\mathfrak{g}]} = ad_a^{-1} ad_b.$$

The functions $\mathcal{A}_a$ are integrals of the Euler equations (for example see [33]):

$$\dot{\xi} = [\xi, \nabla h_{a,b,D}(\xi)], \quad h_{a,b,D}(\xi) = \frac{1}{2}\langle \varphi_{a,b,D}(\xi), \xi \rangle.$$

In the case of compact Lie groups, among the sectional operators there are positive definite ones. Then $H_{a,b,D} = h_{a,b,D} \circ \Phi$ is the Hamiltonian of the geodesic flow of a certain metric that we shall denote by $ds^2_{a,b,D}$.

From theorem 2.1 we obtain the following theorem:

Theorem 4.1 **[8]** *Let Q be a homogeneous space $\mathfrak{G}/\mathfrak{H}$, where $\mathfrak{G}$ is a compact connected Lie group. Then the geodesic flows of the metrics $ds^2_{a,b,D}$ on Q are completely integrable. In particular, for $\varphi_{a,b,D} = Id_{\mathfrak{g}}$ we have integrability of the geodesic flow of the metric ds^2_0.*

Previous results include the case of integrability of geodesic flows on compact Lie groups [22], symmetric spaces [10, 21, 24, 32] and the spaces $\mathfrak{G}/\mathfrak{H}$ where $(\mathfrak{G}, \mathfrak{H})$ form a Gelfand (or spherical) pair [16, 25] (exception are $SO(n)/SO(n-2)$ [32] and $SU(3)/T^2$ [28]). For symmetric spaces and spherical pairs, in a neighborhood of a generic point $x \in T(\mathfrak{G}/\mathfrak{H})$ each $\mathfrak{G}$–invariant function f can be expressed as $f = h \circ \Phi$ and thus we can use just functions from $\mathcal{F}_1$ to get the integrability of any $\mathfrak{G}$–invariant geodesic flow on $\mathfrak{G}/\mathfrak{H}$. Spherical pairs are classified in [19, 25].

Examples given in [6, 7, 11] are homogeneous spaces $\mathfrak{G}/\Gamma$ of noncompact groups, where $\Gamma \subset \mathfrak{G}$ is a discrete cocompact subgroup. The corresponding geodesic flows are integrable by smooth integrals, and by Taimanov's theorem [30, 31] can not be integrable by analytic ones.

5 Bi-quotients of Lie groups

Let $\mathfrak{G}$ be a compact connected Lie group. Consider a subgroup $\mathfrak{U}$ of $\mathfrak{G} \times \mathfrak{G}$ and define the action of $\mathfrak{U}$ on $\mathfrak{G}$ by:

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}, \quad (g_1, g_2) \in \mathfrak{U}, \ g \in \mathfrak{G}.$$

If the action is free then the orbit space $\mathfrak{G}/\mathfrak{U}$ is a smooth manifold called a *bi-quotient* of the Lie group $\mathfrak{G}$. In particular, if $\mathfrak{U} = \mathfrak{K} \times \mathfrak{H}$, where $\mathfrak{K}$ and $\mathfrak{H}$ are subgroups of $\mathfrak{G}$, then the bi-quotient of $\mathfrak{G}$ is denoted by $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$. The condition that $\mathfrak{U} = \mathfrak{K} \times \mathfrak{H}$ acts freely on $\mathfrak{G}$ is:

$$g\mathfrak{K}g^{-1} \cap \mathfrak{H} = e, \quad \text{for any} \ \ g \in \mathfrak{G}.$$

The bi-invariant metric on $\mathfrak{G}$ gives the Riemannian metric on $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$. Denote such metric by ds^2_0.

In this section we shall prove the following general statement:

Theorem 5.1 *Let Q be a bi-quotient $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$, where $\mathfrak{G}$ is a compact connected Lie group. Then the geodesic flow of the metric ds_0^2 on Q induced by a bi-invariant metric on $\mathfrak{G}$ is completely integrable in the non-commutative sence by means of analytic integrals, and therefore in the classical commutative sence by means of C^∞–smooth integrals.*

Proof. We shall use the following description of tangent spaces of $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$. By definition the elements of this bi-quotient space are equivalence classes of the form $\{kgh\}$ where g is fixed, and k and h run over $\mathfrak{K}$ and $\mathfrak{H}$ respectively.

The tangent space for this equivalence class at the point g can be represented in the form

$$g\mathfrak{h} + \mathfrak{k}g, \tag{5.1}$$

so that the tangent space to the bi-quotient at the point $\mathfrak{K}g\mathfrak{H}$ can naturally be represented as the orthogonal complement $(g\mathfrak{h} + \mathfrak{k}g)^\perp$ with respect to the bi-invariant metric on $\mathfrak{G}$. If we want this definition not to depend on the choice of a representative in the class, we must explain how these orthogonal complements are identified at points g and kgh. Since kgh is represented in this form uniquely, the elements k and h are well defined. Then we can consider the map $L_k \circ R_h$ which maps g onto kgh and take its differential. As a result, the subspace $g\mathfrak{h}+\mathfrak{k}g$ is mapped onto $kg\mathfrak{h}h+k\mathfrak{k}gh$, which obviously coincides with $kgh\mathfrak{h}+\mathfrak{k}kgh$. The orthogonal complements will also be matched. Such an identification is natural because the corresponding vectors from the orthogonal complements are projected onto the same vector tangent to the bi-quotient.

The next thing we want to do is to describe some important functions on $T(\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H})$ using their liftings onto $T\mathfrak{G}$. In what follows, we shall denote by $\mathcal{O}_{\mathfrak{K}}(\xi)$ and $\mathcal{O}_{\mathfrak{H}}(\xi)$ the orbits through $\xi \in \mathfrak{g}$ of the natural adjoint actions of subgroups $\mathfrak{K}$ and $\mathfrak{H}$ respectively. All functions will be analytic, polynomial in velocities.

Let f be a function on $T\mathfrak{G}$ which is right-invariant (with respect to $\mathfrak{G}$) and left-invariant with respect to $\mathfrak{K}$ (equivalently, f can be seen as a function obtained by right-translations from some $Ad_{\mathfrak{K}}$ invariant function on $\mathfrak{g}$). Then this function being restricted onto the orthogonal complement is well projected onto $T(\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H})$. Denote the space of functions so obtained by $\mathcal{F}_1$.

Analogously, let f be a function on $T\mathfrak{G}$ which is left-invariant (with respect to $\mathfrak{G}$) and right-invariant with respect to $\mathfrak{H}$. Then this function being restricted onto the orthogonal complement is well projected onto $T(\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H})$. Denote the space of functions so obtained by $\mathcal{F}_2$. Note that $\{\mathcal{F}_1, \mathcal{F}_2\} = 0$.

These functions give the family of first integrals $\mathcal{F}_1 + \mathcal{F}_2$ for the geodesic flow of the metric ds_0^2 on $\mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}$.

The problem is to verify the completeness of this family.

The functions we have constructed have another natural description. Consider the following subset in $\mathfrak{g}$:

$$C = \mathfrak{k}^\perp \cap \{g\mathfrak{h}^\perp g^{-1},\ g \in \mathfrak{G}\} = \cup_{g\in\mathfrak{G}}(\mathfrak{k} + g\mathfrak{h}g^{-1})^\perp \tag{5.2}$$

This set is $Ad_{\mathfrak{K}}$-invariant. So we can consider $\mathfrak{K}$–invariant functions on it. Take such a function and extend it by right translations to the whole of $T\mathfrak{G}$. Actually we shall obtain a function defined for each fiber $T_g\mathfrak{G}$ only on the subset Cg, but this subset contains $(\mathfrak{k}g + g\mathfrak{h})^\perp$ because $(\mathfrak{k}g + g\mathfrak{h})^\perp = (\mathfrak{k} + g\mathfrak{h}g^{-1})^\perp g \subset Cg$. We should remark that C is not smooth in general. But being an algebraic set it is smooth almost everywhere.

By $\mathcal{F}_C$ denote the set of $Ad_{\mathfrak{K}}$ invariant functions on C and by $\operatorname{reg} C$ the set of regular orbits $\mathcal{O}_{\mathfrak{K}}(\xi)$ of $Ad_{\mathfrak{K}}$ action on C.

From the choice of the functions $\mathcal{F}_C$, we have the important relation that the number of functions in $\mathcal{F}_C$ is equal to the number of functions in $\mathcal{F}_1$:

$$\operatorname{ddim} \mathcal{F}_C = \operatorname{ddim} \mathcal{F}_1 \tag{5.3}$$

The number of independent $\mathfrak{K}$–invariant functions on C is equal to the difference:

$$\operatorname{ddim} \mathcal{F}_C = \dim C - \dim \mathcal{O}_{\mathfrak{K}}(\xi), \quad \xi \in \operatorname{reg} C \tag{5.4}$$

On the other hand, the tangent space of C in ξ can be represented as the intersection of $T_\xi\{g\mathfrak{h}^\perp g^{-1},\ g \in \mathfrak{G}\}$ and $\mathfrak{k}^\perp$. The tangent plane of the set $\{g\mathfrak{h}^\perp g^{-1},\ g \in \mathfrak{G}\}$ at a generic point ξ can be obviously presented as $\mathfrak{h}^\perp + [\xi, \mathfrak{g}]$. Thus:

$$T_\xi C = (\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) \cap \mathfrak{k}^\perp. \tag{5.5}$$

Also notice that $\dim \mathcal{O}_{\mathfrak{K}}(\xi) = \dim[\xi, \mathfrak{k}]$. Thus, by (5.5) the number of $\mathfrak{K}$-invariant functions is equal to

$$\dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) \cap \mathfrak{k}^\perp - \dim[\xi, \mathfrak{k}]. \tag{5.6}$$

Let us compute this number. We have

$$\begin{aligned}
&\dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) \cap \mathfrak{k}^\perp = \\
&\dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) + \dim \mathfrak{k}^\perp - \dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}] + \mathfrak{k}^\perp) = \\
&(\text{we use that } \mathfrak{h}^\perp + \mathfrak{k}^\perp = \mathfrak{g}) \\
&\dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) - \dim \mathfrak{k}
\end{aligned}$$

Next,

$$\begin{aligned}
&\dim[\xi, \mathfrak{k}] = \dim \mathfrak{k} - \dim \mathfrak{k}_\xi = \\
&\dim \mathfrak{k} - \dim(\mathfrak{g}_\xi \cap \mathfrak{k}) = \\
&\dim \mathfrak{k} - \dim([\xi, \mathfrak{g}]^\perp \cap \mathfrak{k}) = \\
&\dim \mathfrak{k} - \dim \mathfrak{g} + \dim([\xi, \mathfrak{g}] + \mathfrak{k}^\perp),
\end{aligned}$$

where $\mathfrak{k}_\xi = \{\eta \in \mathfrak{k},\ [\eta, \xi] = 0\}$, $\mathfrak{g}_\xi = \{\eta \in \mathfrak{g},\ [\eta, \xi] = 0\}$.

Thus, by (5.3) and (5.6) we get:

$$\begin{aligned}
\operatorname{ddim} \mathcal{F}_1 &= \dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) - \dim \mathfrak{k} - \dim \mathfrak{k} + \dim \mathfrak{g} - \dim([\xi, \mathfrak{g}] + \mathfrak{k}^\perp) = \\
&= \dim \mathfrak{g} - 2\dim \mathfrak{k} + \dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) - \dim([\xi, \mathfrak{g}] + \mathfrak{k}^\perp),
\end{aligned}$$

for ξ that belongs to the open dense subset $\operatorname{reg} C \subset C$.

Now one can notice that the result for the number of functions in the family $\mathcal{F}_2$ will be just the same (after interchanging $\mathfrak{k}$ and $\mathfrak{h}$).

Finally, we see that

$$\begin{aligned}
&\operatorname{ddim} \mathcal{F}_1 + \operatorname{ddim} \mathcal{F}_2 = \\
&\dim \mathfrak{g} - 2\dim \mathfrak{k} + \dim(\mathfrak{h}^\perp + [\xi, \mathfrak{g}]) - \dim([\xi, \mathfrak{g}] + \mathfrak{k}^\perp) + \\
&\dim \mathfrak{g} - 2\dim \mathfrak{h} + \dim(\mathfrak{k}^\perp + [\xi, \mathfrak{g}]) - \dim([\xi, \mathfrak{g}] + \mathfrak{h}^\perp) = \\
&2(\dim \mathfrak{g} - \dim \mathfrak{k} - \dim \mathfrak{h}) = 2\dim \mathfrak{K}\backslash\mathfrak{G}/\mathfrak{H}.
\end{aligned}$$

The theorem follows from the lemma below. q.e.d.

Lemma 5.1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Poisson subalgebras of $(C^\infty(M), \{\cdot, \cdot\}_M)$ on a symplectic manifold M that commute, i.e., $\{\mathcal{F}_1, \mathcal{F}_2\} = 0$. If*

$$\operatorname{ddim} \mathcal{F}_1 + \operatorname{ddim} \mathcal{F}_2 = \dim M,$$

then $\mathcal{F}_1 + \mathcal{F}_2$ is a complete algebra of functions on M.

Proof. Let x belong to $\operatorname{reg} \mathcal{F}_1 \cap \operatorname{reg} \mathcal{F}_2$. Consider linear spaces:

$$\begin{aligned}
W_1 = W_1(x) = \operatorname{span}\{\operatorname{sgrad} f(x),\ f \in \mathcal{F}_1\} \subset T_x M, \\
W_2 = W_2(x) = \operatorname{span}\{\operatorname{sgrad} f(x),\ f \in \mathcal{F}_2\} \subset T_x M.
\end{aligned}$$

Under the suppositions of the lemma, we have:

$$\dim W_1 + \dim W_2 = \dim M, \quad \omega(W_1, W_2) = 0. \tag{5.7}$$

From $\omega(W_1, W_2) = 0$ it follows that symplectic orthogonal complement of $W_1 + W_2$ contains $W_1 \cap W_2$: $W_1 \cap W_2 \subset (W_1 + W_2)^\omega$. On the other hand, (5.7) implies:

$$\dim(W_1 \cap W_2) = \dim(W_1 + W_2)^\omega = \dim M - \dim(W_1 + W_2)$$

Therefore, $W_1 \cap W_2 = (W_1 + W_2)^\omega \subset W_1 + W_2$ and by remark 1.1, $\mathcal{F}_1 + \mathcal{F}_2$ is a complete algebra of functions on M. q.e.d.

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