Normal forms for pseudo-Riemannian 2-dimensional metrics whose geodesic flows admit integrals quadratic in momenta

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Abstract

We discuss pseudo-Riemannian metrics on 2-dimensional manifolds such that the geodesic flow admits a nontrivial integral quadratic in velocities. We construct (Theorem 1) local normal forms of such metrics. We show that these metrics have certain useful properties similar to those of Riemannian Liouville metrics, namely:

• they admit geodesically equivalent metrics (Theorem 2);

• one can use them to construct a large family of natural systems admitting integrals quadratic in momenta (Theorem 4);

• the integrability of such systems can be generalized to the quantum setting (Theorem 5);

• these natural systems are integrable by quadratures (Section 2.2.2).

1 Introduction

Consider a pseudo-Riemannian metric $g = (g_{ij})$ on a surface M^2. A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2}g^{ij}p_ip_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric. Geometrically, this condition means that the function is constant on the orbits of the Hamiltonian system with the Hamiltonian H. We say the integral F is **quadratic in momenta** if, in every local coordinate system (x, y) on M^2, it has the form

$$a(x, y)p_x^2 + b(x, y)p_xp_y + c(x, y)p_y^2, \tag{1}$$

with (x, y, p_x, p_y) canonical coordinates on T^*M^2. Geometrically, formula (1) means that the restriction of the integral to every cotangent space $T_p^*M^2 \equiv \mathbb{R}^2$ is a homogeneous quadratic function. Of course, H itself is an integral quadratic in momenta for g. We will say that the integral F is **nontrivial**, if $F \neq \text{const} \cdot H$ for all const $\in \mathbb{R}$.

The main result of this paper is Theorem 1 below, which gives us a list of local normal forms of metrics of signature $(+, -)$ whose geodesic flows admit a nontrivial integral quadratic in momenta. For the Riemannian case (and, therefore, for the signature $(-, -)$) such metrics are the well-known Liouville metrics.

Theorem 1. *Suppose the metric g of signature $(+, -)$ on M^2 admits a nontrivial integral quadratic in momenta. Then, in a neighbourhood of almost every point there exist coordinates x, y such that the metric and the integral are as in the following table:*

	Liouville case	*Complex-Liouville case*	*Jordan-block case*
g	$(X(x) - Y(y))(dx^2 - dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\,dxdy$
F	$\frac{X(x)p_y^2 - Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_xp_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_xp_y$

where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x+i\cdot y$.

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Given a metric and the quadratic integral, it is easy to understand what case they belong to. Indeed, for the integral (1) the matrix

$$F^{ij} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

can be viewed as a $(2,0)$-tensor: if we change the coordinate system and rewrite the function F in the new coordinates, the matrix changes according to the tensor rule. Then,

$$G^i_j := \sum_\alpha g_{j\alpha} F^{i\alpha} \tag{2}$$

is a $(1,1)$-tensor. By direct calculation we see that G^i_j has two different real eigenvalues in the first case, two complex-conjugate eigenvalues in the second case and is (conjugate to) a Jordan-block in the third case. This also explains our choice of the names for the normal forms of the metrics. Indeed, in the Riemannian case, the tensor (2) always has two real eigenvalues. In particular, the normal form of the Riemannian metric admitting an integral quadratic in momenta, which is traditionally called Liouville form (or Liouville metric), is very similar to the metric of our "Liouville" case. One can view our "Complex-Liouville" case as the complexification of the standard Liouville metric: if in the expression

$$(X(x) - Y(y))(dx^2 + dy^2)$$

we replace X by (a holomorphic function) $h(z)$, Y by $\overline{h(z)}$, dx by dz, and dy by $id\bar{z}$, we obtain the Complex-Liouville metric up to the factor $8i$. The Jordan-block case has no direct analog in the Riemannian setting.

Remark 1. The corresponding natural Hamiltonian problem on the hyperbolic plane has recently been treated in [39] following an approach used by Rosquist and Uggla [40]. Systems with indefinite signature have been investigated before in the classical works by Kalnins and Miller on separation of variables [20, 38], see also [13, 16, 36]. Other possible approaches are based on Killing tensor theory [5], r-matrix theory [24] and algebraic methods [15]. For the corresponding quantum case we refer to [19] and references therein.

Remark 2. A part, if not all credits for the results of the present paper should be given to Darboux, see [14, §§592–594,600–608]. There is no doubt that Darboux was very close to Theorem 1, to the results of Section 2.2.2, and, to a certain extent, to Theorem 2 of our paper, and could get it if he would have been interested in the pseudo-Riemannain metrics. More precisely,

- In [14, §593], Darboux gets the Riemannian Liouville metrics. Since he worked over complex coordinates, his formulas can be interpreted as our Liouville and Complex-Liouville cases.
- In [14, §594], Darboux gets (a case that could be interpreted as) the Jordan-block case.
- The formulas of Section 2.2.2 of the present paper are similar to that of [14, §594].

However, Darboux was interested in the positive definite metrics only. Actually, in his time it was unusual to consider indefinite metrics, since the applications of pseudo-Riemannian metrics to general relativity and cosmology appeared much later. Darboux worked over complex coordinates x, y and explicitly remarked on the transformation $x = u + iv, y = u - iv$ leading to the standard metric of the (+,+) case, with no mention of a possible interpretation of x, y as real coordinates. The only exception is the Jordan-block case with constant function Y (equations (24,25) of [14, §594]), where one can get the surfaces of revolution.

The results of this paper were announced in [10].

2 Applications

2.1 Applications in geometry: normal forms for 2-dimensional geodesically equivalent metrics

Two metrics g and $\bar{g}$ on one manifold are **geodesically equivalent,** if every (unparametrized) geodesic of the first metric is a geodesic of the second metric. Investigation of geodesically equivalent metrics is a classical problem in differential geometry, see the surveys [3, 33, 37] or/and the introductions to [31, 32, 34]. In particular, normal forms for geodesically equivalent Riemannian 2-dimensional metrics were already constructed by Dini [17]. An easy corollary of Theorem 1 is the following theorem which gives normal forms of geodesically equivalent nonproportional metrics such that one of them has signature $(+,-)$.

Theorem 2. *Let g, $\bar g$ be geodesically equivalent metrics on M^2 such that g has signature $(+,-)$, and $\bar g \neq const \cdot g$ for every $const \in \mathbb{R}$. Then, in a neighbourhood of almost every point, there exist coordinates such that metrics are as in the following table:*

	Liouville case	*Complex-Liouville case*	*Jordan-block case*
g	$(X(x)-Y(y))(dx^2-dy^2)$	$\Im(h)dxdy$	$(1+xY'(y))\,dxdy$
$\bar g$	$\left(\frac{1}{Y(y)}-\frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)}-\frac{dy^2}{Y(y)}\right)$	$-\left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2 dx^2 + 2\frac{\Re(h)\Im(h)}{(\Im(h)^2+\Re(h)^2)^2}dxdy + \left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2 dy^2$	$\frac{1+xY'(y)}{Y(y)^4}\left(-2Y(y)dxdy + (1+xY'(y))dy^2\right)$

where h is holomorphic function of the variable $z := x+i\cdot y$.

Remark 3. It it natural to consider the metrics from the Complex-Liouville case as the complexification of the metrics from the Liouville case: indeed, in the complex coordinates $z = x+i\cdot y$, $\bar z = x-i\cdot y$, the metrics have the form

$$\begin{aligned} ds_g^2 &= -\tfrac{1}{8}(\overline{h(z)}-h(z))\left(d\bar z^2 - dz^2\right), \\ ds_{\bar g}^2 &= -\tfrac{1}{4}\left(\frac{1}{\overline{h(z)}}-\frac{1}{h(z)}\right)\left(\frac{d\bar z^2}{\overline{h(z)}}-\frac{dz^2}{h(z)}\right). \end{aligned}$$

Remark 4. In the Jordan-block case, if $dY \neq 0$ (which is always the case at almost every point, if the restriction of g to any neighborhood does not admit a Killing vector field), after a local coordinate change, the metrics g and $\bar g$ have the form (see also Remark 14)

$$\begin{aligned} ds_g^2 &= \left(\tilde Y(y)+x\right)dxdy \\ ds_{\bar g}^2 &= -\frac{2(\tilde Y(y)+x)}{y^3}dxdy + \frac{(\tilde Y(y)+x)^2}{y^4}dy^2. \end{aligned}$$

Proof of Theorem 1. We will use the next theorem which probably was already known to Darboux [14, §608]. For recent proofs, see [26, 27, 28, 44].

Theorem 3. *Let g be a metric on M^2 and $h \in \Gamma(S_2M^2)$ be a symmetric nondegenerate bilinear form on M^2. Consider the following metric*

$$\bar g = \left(\frac{\det(g)}{\det(h)}\right)^2 h \tag{3}$$

on M^2. If g and $\bar g$ are geodesically equivalent, then the function

$$\hat h : TM \to \mathbb{R},\ \hat h(\xi) := h(\xi,\xi)$$

is an integral for the geodesic flow of g.

Remark 5. Theorem 3 and Corollary 1 below bear some resemblance with other classes of transformations between dynamical systems [1, 2, 11, 38, 41, 42, 43]. However, the present result is of different nature and is deeper because, in order to construct the second system, one needs to know the quadratic integral of the first one.

Combining Theorem 3 with Theorem 1, we obtain that, in a neighbourhood of almost every point, geodesically equivalent metrics g and $\bar g$ are as in the table in Theorem 2 (we assume that g has signature $(+,-)$ and that $\bar g \neq \text{const}\cdot g$). Thus, in order to prove Theorem 2, we need to show that the metrics from the table are indeed geodesically equivalent, which can be done by direct calculations. Indeed, it is well-known, see for example [18, §40 of Ch. III], that two metrics are geodesically equivalent if and only if the difference of their Levi-Cìvita connections has the form $\Upsilon_j \delta^i_k + \Upsilon_k \delta^i_j$ for a one-form $\Upsilon = (\Upsilon_i)$. Direct calculation of the Levi-Cìvita connections for the metrics shows that it is indeed the case: the form Υ equals

$$\frac{1}{2}\left(\frac{X'(x)}{X(x)}dx + \frac{Y'(y)}{Y(y)}dy\right)$$

for the normal forms of the metrics in the Liouville case,

$$\frac{\Im(h)\frac{\partial}{\partial x}\Im(h) + \Re(h)\frac{\partial}{\partial y}\Im(h)}{\left(\Im(h)\right)^2 + \left(\Re(h)\right)^2}dx + \frac{\Im(h)\frac{\partial}{\partial y}\Im(h) - \Re(h)\frac{\partial}{\partial x}\Im(h)}{\left(\Im(h)\right)^2 + \left(\Re(h)\right)^2}dy$$

for the complex Liouville case and $\frac{Y'(y)}{Y(y)}dy$ for the Jordan-block case. ☐

Corollary 1. *Let g be a metric on M^2 and $h \in \Gamma(S_2 M^2)$ be a symmetric nondegenerate bilinear form on M^2. Then, g and the metric* (3) *are geodesically equivalent, if and only if the function*

$$\hat{h} : TM \to \mathbb{R},\ \hat{h}(\xi) = h(\xi, \xi)$$

is an integral for the geodesic flow of g.

Proof. In the direction "$\Longrightarrow$" the statement coincides with Theorem 3. In order to prove in "$\Longleftarrow$" direction, it is sufficient to check the statement in the neighbourhood of almost every point. Here, the metrics g, $\bar{g}$ and the integrals $\hat{h}$ are given by Theorems 1,2 and are related precisely by formula (3). ☐

Remark 6. Theorem 3 had found a recent important application in the solution of two problems explicitly stated by Sophus Lie in [25] due to [12, 35].

2.2 Applications in mathematical physics

2.2.1 Natural systems admitting an integral quadratic in momenta

For a pseudo-Riemannian manifold (M, g), a **natural Hamiltonian system** is a Hamitonian system with $H : T^*M \to \mathbb{R}$ of the form $H := H_g + U = \frac{1}{2}g^{ij}p_i p_j + U(x, y)$. We say that a natural Hamiltonian system is **quadratically integrable**, if there exists a function F of the form $F = F_g + V = F^{ij}p_i p_j + V(x, y)$ such that $\{H, F\} = 0$ with $F \neq \text{const}_1 \cdot H + \text{const}_2$ for all $\text{const}_1, \text{const}_2 \in \mathbb{R}$.

Remark 7. In [39], the natural Hamiltonian system on the Minkowski plane has been reduced to the corresponding kinetic Hamiltonian system with conformal (Jacobi) pseudo-Euclidean metric.

Theorem 4. *Let g be a metric of signature $(+,-)$ on M^2. Assume a natural Hamiltonian system with Hamiltonian $H_g + U$ to be quadratically integrable with integral $F = F_g + V$. Then, in a neighbourhood of almost every point, there exists a coordinate system such that the metric g and the functions F_g, U, V are as in the following table:*

	Liouville case	*Complex-Liouville case*	*Jordan-block case*
g	$(X(x) - Y(y))(dx^2 - dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\,dxdy$
F_g	$\frac{X(x)p_y^2 - Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y,$	$p_x^2 - 2\frac{Y(y)}{1 + xY'(y)}p_x p_y$
U	$\frac{1}{2}\frac{\hat{X}(x) - \hat{Y}(y)}{X(x) - Y(y)}$	$\frac{\Im(h_1)}{\Im(h)}$	$\frac{xY_1'(y) + Y_2(y)}{1 + xY'(y)}$
V	$\frac{\hat{Y}(y)X(x) - \hat{X}(x)Y(y)}{X(x) - Y(y)}$	$\Re(h)\frac{\Im(h_1)}{\Im(h)} - \Re(h_1)$	$-Y\frac{xY_1'(y) + Y_2(y)}{1 + xY'(y)} + Y_1(y)$

where h, h_1 are holomorphic functions of the variable $z := x + i \cdot y$.

Proof. It is well known (see, for example, [6]), that the condition $\{H, F\} = 0$ is in this case equivalent to the following two conditions:

$$\{H_g, F_g\} = 0\,, \tag{4}$$
$$2dU \circ G = dV\,, \tag{5}$$

where G is given by (2). In tensor index notations, (5) is

$$2G^i_j \frac{\partial U}{\partial x^i} = \frac{\partial V}{\partial x^j}\,. \tag{6}$$

Indeed, condition $\{H, F\} = 0$ is equivalent to the following equation:

$$\{H_g, F_g\} + \{H_g, V\} - \{F_g, U\} = 0.$$

Since $\{H_g, F_g\}$ (respectively, $\{H_g, V\} - \{F_g, U\}$) is a third degree-polynomial in momenta (respectively, first degree), the latter equation is equivalent to:

$$\begin{aligned} \{H_g, F_g\} &= 0 && (7)\\ \{F_g, U\} &= \{H_g, V\}\,. && (8) \end{aligned}$$

We see that (7) coincides with (4) and (8) is equivalent to

$$2F^{ij}\frac{\partial U}{\partial x^i} = g^{ij}\frac{\partial V}{\partial x^i}\,,$$

which is equivalent to (6) and therefore to (5).

Condition (4) tells us that the function F_g is an integral quadratic in momenta for the geodesic flow of g. Clearly, F_g is nontrivial. Indeed, if $F_g = \mathrm{const}_1 \cdot H_g$, then condition (5) reads $\mathrm{const}_1 \circ dU = dV$ implying $V = \mathrm{const}_1 \cdot U + \mathrm{const}_2$. These in turn imply $F = \mathrm{const}_1 \cdot H + \mathrm{const}_2$, which contradicts the assumptions.

Thus, F_g is a nontrivial integral of the geodesic flow of the metric g. By Theorem 1, almost every point has a neighbourhood with local coordinates (x, y) such that g and F_g are as in the table. In order to prove Theorem 4, it is sufficient to show that, for every column of the table, the functions U and V are complete solutions of equation (5). Here we consider the three cases in detail.

Liouville case. Assume g, F_g are as in the first column of the table. Then the form $dU \circ G$ is

$$-Y(y)\frac{\partial U}{\partial x}dx - X(x)\frac{\partial U}{\partial y}dy$$

and condition (5) reads

$$\left\{\begin{array}{rcl} \frac{\partial Y(y)U}{\partial x} &=& -\frac{1}{2}\frac{\partial V}{\partial x}\,,\\ \frac{\partial X(x)U}{\partial y} &=& -\frac{1}{2}\frac{\partial V}{\partial y}\,. \end{array}\right. \tag{9}$$

Differentiating the second equation w.r.t. x and subtracting the derivative of the first equation w.r.t. y, we obtain

$$0 = \frac{\partial}{\partial x}\left(X(x)\frac{\partial U}{\partial y}\right) - \frac{\partial}{\partial y}\left(Y(y)\frac{\partial U}{\partial x}\right) = \frac{\partial^2 (X(x) - Y(y))U}{\partial x \partial y}$$

implying

$$U = \tfrac{1}{2}\frac{\hat{X}(x) - \hat{Y}(y)}{X(x) - Y(y)}$$

for certain functions $\hat{X} = \hat{X}(x)$ and $\hat{Y} = \hat{Y}(y)$. Substituting U in (9), we obtain

$$V = \frac{X(x)\hat{Y}(y) - Y(y)\hat{X}(x)}{X(x) - Y(y)}\,.$$

Thus, in the Liouville case, U and V are as in the table.

Complex-Liouville case. In this case $2dU \circ G$ is equal to

$$\begin{aligned} &\left(\Re(h)\frac{\partial U}{\partial x} - \Im(h)\frac{\partial U}{\partial y}\right)dx + \left(\Im(h)\frac{\partial U}{\partial x} + \Re(h)\frac{\partial U}{\partial y}\right)dy\\ =\;& \left(\frac{\partial \Re(h)U}{\partial x} - \frac{\partial \Im(h)U}{\partial y}\right)dx + \left(\frac{\partial \Re(h)U}{\partial y} + \frac{\partial \Im(h)U}{\partial x}\right)dy \end{aligned}$$

and condition (5) is equivalent to the following system of PDE:

$$\begin{cases} \frac{\partial \Re(h)U}{\partial x} - \frac{\partial \Im(h)U}{\partial y} &= \frac{\partial V}{\partial x}, \\ \frac{\partial \Re(h)U}{\partial y} + \frac{\partial \Im(h)U}{\partial x} &= \frac{\partial V}{\partial y}. \end{cases} \tag{10}$$

We see that these equation are precisely the Cauchy-Riemann condition for the function $h_1 := \Re(h)U - V + i \cdot \Im(h)U$. Thus,

$$U = \frac{\Im(h_1)}{\Im(h)}$$

and

$$V = \Re(h)U - \Re(h_1) = \Re(h)\frac{\Im(h_1)}{\Im(h)} - \Re(h_1).$$

We see that U and V are as in the table.

Jordan-block case. In this case the 1-form $2dU \circ G$ is

$$-Y(y)\frac{\partial U}{\partial x}dx + \left((1 + xY'(y))\frac{\partial U}{\partial x} - Y(y)\frac{\partial U}{\partial y}\right)dy$$

and condition (5) is equivalent to the following system of PDE:

$$\begin{cases} -Y(y)\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial x}, \\ (1 + xY'(y))\frac{\partial U}{\partial x} - Y(y)\frac{\partial U}{\partial y} &= \frac{\partial V}{\partial y}. \end{cases} \tag{11}$$

The first equation in (11) is equivalent to $V = -Y(y)U + Y_1(y)$. Substituting this in the second equation, we obtain

$$(1 + xY'(y))\frac{\partial U}{\partial x} - Y(y)\frac{\partial U}{\partial y} = -\frac{\partial Y(y)U}{\partial y} + Y_1'(y)$$

which implies

$$\frac{\partial(1 + xY'(y))U}{\partial x} = Y_1'(y)$$

and therefore $(1 + xY'(y))U = xY_1'(y) + Y_2(y)$. Thus,

$$U = \frac{xY_1'(y) + Y_2(y)}{1 + xY'(y)}$$

and

$$V = -Y\frac{xY_1'(y) + Y_2(y)}{1 + xY'(y)} + Y_1(y).$$

□

2.2.2 Integration by quadratures of natural systems admitting an integral quadratic in momenta

Since the time of Jacobi it is known that (in the 2-dimensional Riemannian case) nontrivial integrals quadratic in momenta are extremely helpful for the description of dynamics of natural systems: indeed, in this case

- the Hamilton equations, which are a system of four ODE on T^*M^2, can be reduced to a parameter-depending system of two ODE on M^2.
- Moreover, it is possible to construct a **characteristic** (= function constant on the solutions) of this system by means of the integration of certain functions of one variable only.

See [7, 45] for details.

Classically, the second property is referred to as **"the system is integrable by quadratures".** Both properties are useful for exact solutions, for numerical analysis and for a qualitative description of (the solutions of) the Hamilton equations. We are going to show that these nice properties persist in the pseudo-Riemannian setting.

Liouville case. There is virtually no difference with respect to the Riemannian setting. Consider $H = H_g + U$ and $F = F_g + V$ such that g, F_g, U, V are as in the first column of the table from Theorem 4. Then, the first two Hamilton equations are

$$\left\{\begin{array}{rclcl} \frac{d}{dt}x & = & \frac{\partial H}{\partial p_x} & = & \frac{p_x}{X-Y}\,, \\ \frac{d}{dt}y & = & \frac{\partial H}{\partial p_y} & = & -\frac{p_y}{X-Y}\,. \end{array}\right. \tag{12}$$

Since the functions F and H are constant on the solutions of the system, for every point (x, y, p_x, p_y) of the solution we have

$$\left\{\begin{array}{rcl} \frac{1}{2}\frac{p_x^2-p_y^2}{X(x)-Y(y)} + \frac{1}{2}\frac{\hat X(x)-\hat Y(y)}{X(x)-Y(y)} & = & H_0\,, \\ \frac{X(x)p_y^2-Y(y)p_x^2}{X(x)-Y(y)} + \frac{\hat Y(y)X(x)-\hat X(x)Y(y)}{X(x)-Y(y)} & = & F_0\,. \end{array}\right.$$

This is a linear system on p_x^2, p_y^2, solving it w.r.t. p_x and p_y we obtain

$$\left\{\begin{array}{rcl} p_x^2 & = & 2H_0X(x) + F_0 - \hat X(x)\,, \\ p_y^2 & = & 2H_0Y(y) + F_0 - \hat Y(y)\,. \end{array}\right. \tag{13}$$

Substituting these in (12), we obtain

$$\left\{\begin{array}{rclcl} \frac{d}{dt}x & = & \varepsilon_1\frac{\sqrt{2H_0X(x)+F_0-\hat X(x)}}{X-Y} & := & v_1\,, \\ \frac{d}{dt}y & = & \varepsilon_2\frac{\sqrt{2H_0Y(y)+F_0-\hat Y(y)}}{X-Y} & := & v_2\,. \end{array}\right. \tag{14}$$

We see that Hamilton equations can be reduced to a system of two ODE on M^2 depending on the parameters $H_0, F_0 \in \mathbb{R}$ and $\varepsilon_i \in \{-1, +1\}$.

Clearly, a function $K(x, y)$ is a characteristic of the system (14) if dK vanishes on the vector field $v := (v_1, v_2)$. Since the form

$$B := \frac{\varepsilon_1 dx}{\sqrt{2H_0X(x) + F_0 - \hat X(x)}} - \frac{\varepsilon_2 dy}{\sqrt{2H_0Y(y) + F_0 - \hat Y(y)}}$$

vanishes on v and is closed, the function

$$K(p) := \int_{p_0}^{p} B = \int_{x_0}^{x} \frac{d\xi}{\sqrt{2H_0X(\xi) + F_0 - \hat X(\xi)}} - \varepsilon_1\varepsilon_2 \int_{y_0}^{y} \frac{d\xi}{\sqrt{2H_0Y(\xi) + F_0 - \hat Y(\xi)}}$$

is a characteristic. We see that in order to find a characteristic, we only need to integrate two functions of one variable each, i.e., the system is integrable by quadratures.

Complex-Liouville case. Consider $H = H_g + U$ and $F = F_g + V$ such that g, F_g, U, V are as in the second column of the table from Theorem 4. Then, the first two Hamilton equations are

$$\left\{\begin{array}{rclcl} \frac{d}{dt}x & = & \frac{\partial H}{\partial p_x} & = & \frac{2p_y}{\Im(h)}\,, \\ \frac{d}{dt}y & = & \frac{\partial H}{\partial p_y} & = & \frac{2p_x}{\Im(h)}\,. \end{array}\right. \tag{15}$$

Since the functions F and H are constant on the solutions of the system, for every point (x, y, p_x, p_y) of the solution we have

$$\left\{\begin{array}{rcl} 2\frac{p_xp_y}{\Im(h)} + \frac{\Im(h_1)}{\Im(h)} & = & H_0\,, \\ p_x^2 - p_y^2 + \Re(h)\left(2\frac{p_xp_y}{\Im(h)} + \frac{\Im(h_1)}{\Im(h)}\right) - \Re(h_1) & = & F_0\,. \end{array}\right.$$

Subtracting the first equation times $\Re(h)$ from the second, we obtain

$$\left\{\begin{array}{rcl} 2p_xp_y &=& H_0\Im(h)-\Im(h_1)\,, \\ p_x^2-p_y^2 &=& -\left(\Re(h)H_0-\Re(h_1)\right)+F_0\,. \end{array}\right.$$

From these, adding (respectively, substracting) to (respectively, from) the second equation the first equation times i, we obtain

$$\left\{\begin{array}{rclcl} (p_x-i\cdot p_y)^2 &=& -\left(H_0\Re(h)-\Re(h_1)-F_0\right)-i\cdot\left(H_0\Im(h)-\Im(h_1)\right) &=& -H_0h+h_1+F_0\,, \\ (p_x+i\cdot p_y)^2 &=& -\left(H_0\Re(h)-\Re(h_1)-F_0\right)+i\cdot\left(H_0\Im(h)-\Im(h_1)\right) &=& -H_0\bar h+\bar h_1+F_0\,. \end{array}\right.$$

Remark 8. Since $\frac{1}{2}(p_x-i\cdot p_y)$ is the canonical momentum conjugate to $z=x+i\cdot y$, these equations are the complex analog of (13).

Then, $p_x=\varepsilon\Re\left(\sqrt{-H_0h+h_1+F_0}\right)$ and $p_y=-\varepsilon\Im\left(\sqrt{-H_0h+h_1+F_0}\right)$ (the choice of the branch of the square root is hidden in ε). Substituting these in (15), we obtain

$$\left\{\begin{array}{rclcl} \frac{d}{dt}x &=& \frac{-2\varepsilon\Im\left(\sqrt{-H_0h+h_1+F_0}\right)}{\Im(h)} &:=& v_1\,, \\ \frac{d}{dt}y &=& \frac{2\varepsilon\Re\left(\sqrt{-H_0h+h_1+F_0}\right)}{\Im(h)} &:=& v_2\,. \end{array}\right. \tag{16}$$

We see that Hamilton equations can be reduced to a system of two ODE on M^2 depending on the parameters $H_0,F_0\in\mathbb{R}$, and $\varepsilon\in\{-1,+1\}$.

Consider the 1-form

$$B:=\frac{\Re\left(\sqrt{-H_0h+h_1+F_0}\right)}{|-H_0h+h_1+F_0|}dx+\frac{\Im\left(\sqrt{-H_0h+h_1+F_0}\right)}{|-H_0h+h_1+F_0|}dy\,.$$

The Cauchy-Riemann conditions for the holomorphic function $\sqrt{-H_0h+h_1+F_0}$ imply that the form is closed. Clearly, the form vanishes on the vector field $v=(v_1,v_2)$. Then, the function

$$K(p):=\int_{p_0}^{p}B=\int_{x_0}^{x}\frac{\Re\left(\sqrt{-H_0h+h_1+F_0}\right)}{|-H_0h+h_1+F_0|}d\xi+\int_{y_0}^{y}\frac{\Im\left(\sqrt{-H_0h+h_1+F_0}\right)}{|-H_0h+h_1+F_0|}d\xi$$

is constant on the solutions of (16), i.e., is a characteristic of the system. It is easy to check by direct calculations that in the complex coordinate z the form B is

$$2\Re\left(\frac{dz}{\sqrt{-H_0h+h_1+F_0}}\right)\,.$$

Thus, the function K equals to

$$2\Re\left(\int_{z_0}^{z}\frac{d\xi}{\sqrt{-H_0h(\xi)+h_1(\xi)+F_0}}\right)\,,$$

i.e., the system is integrable by quadratures.

Jordan-block case. Consider $H=H_g+U$ and $F=F_g+V$ such that g,F_g,U,V are as in the third column of the table from Theorem 4. Then, the first two Hamilton equations are

$$\left\{\begin{array}{rclcl} \frac{d}{dt}x &=& \frac{\partial H}{\partial p_x} &=& \frac{2p_y}{1+xY'(y)}\,, \\ \frac{d}{dt}y &=& \frac{\partial H}{\partial p_y} &=& \frac{2p_x}{1+xY'(y)}\,. \end{array}\right. \tag{17}$$

Since the functions F and H are constant on the solutions of the system, for every point (x,y,p_x,p_y) of the solution we have

$$\left\{\begin{array}{rcl} 2\frac{p_xp_y}{1+xY'(y)}+\frac{Y_2(y)+xY_1'(y)}{1+xY'(y)} &=& H_0\,, \\ p_x^2-Y(y)\left(2\frac{p_xp_y}{1+xY'(y)}+\frac{Y_2(y)+xY_1'(y)}{1+xY'(y)}\right)+Y_1(y) &=& F_0\,. \end{array}\right.$$

Adding the first equation times $Y(y)$ to the second one, we obtain

$$\left\{\begin{array}{lcl} p_x^2 & = & H_0 Y(y) - Y_1(y) + F_0 \\ 2p_x p_y & = & x\left(H_0 Y'(y) - Y_1'(y)\right) + H_0 - Y_2(y) \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} p_x & = & \varepsilon\sqrt{H_0 Y(y) - Y_1(y) + F_0}\,, \\ p_y & = & \frac{\varepsilon}{2}\frac{x\left(H_0 Y'(y) - Y_1'(y)\right) + H_0 - Y_2(y)}{\sqrt{H_0 Y(y) - Y_1(y) + F_0}}\,, \end{array}\right.$$

where $\varepsilon \in \{-1, +1\}$. Substituting these in (17), we obtain

$$\left\{\begin{array}{lclcl} \frac{d}{dt}x & = & \varepsilon\frac{x\left(H_0 Y'(y) - Y_1'(y)\right) + H_0 - Y_2(y)}{(1 + xY'(y))\sqrt{H_0 Y(y) - Y_1(y) + F_0}} & := & v_1\,, \\ \frac{d}{dt}y & = & \varepsilon\frac{2\sqrt{H_0 Y(y) - Y_1(y) + F_0}}{1 + xY'(y)} & := & v_2\,. \end{array}\right. \tag{18}$$

We see that Hamilton equations can be reduced to a system of two ODE on M^2 depending on the parameters $H_0, F_0 \in \mathbb{R}$, and $\varepsilon \in \{-1, +1\}$.

Consider the 1-form

$$B := \frac{dx}{\sqrt{H_0 Y(y) - Y_1(y) + F_0}} - \frac{1}{2}\frac{x\left(H_0 Y'(y) - Y_1'(y)\right) - Y_2(y) + H_0}{\left(H_0 Y(y) - Y_1(y) + F_0\right)^{3/2}}dy \tag{19}$$

$$= d\left[\frac{x}{\sqrt{H_0 Y(y) - Y_1(y) + F_0}}\right] + \frac{1}{2}\frac{Y_2(y) - H_0}{\left(H_0 Y(y) - Y_1(y) + F_0\right)^{3/2}}dy\,. \tag{20}$$

By (20), the form is closed. By (19), the form vanishes on the vector field $v = (v_1, v_2)$. Then, the function

$$K(p) := \int_{p_0}^{p} B = \left.\frac{x}{\sqrt{F_0 - Y_1(y) + H_0 Y(y)}}\right|_{p_0}^{p} + \frac{1}{2}\int_{y_0}^{y}\frac{Y_2(\xi) - H_0}{\left(F_0 - Y_1(\xi) + H_0 Y(\xi)\right)^{3/2}}d\xi$$

is a characteristic of the system (18), i.e. the system is integrable by quadratures.

2.2.3 Quantum integrability

Let g be a metric, and $(F^{ij}) \in \Gamma(S_2 M^2)$ be a symmetric bilinear 2-form on T^*M^2. Consider the following two linear partial differential operators $\Delta_g, \mathcal{F}_g : C^\infty \to C^\infty$:

$$\Delta_g := -\sum_{i,j}\frac{1}{\sqrt{|det(g)|}}\frac{\partial}{\partial x_i}g^{ij}\sqrt{|det(g)|}\frac{\partial}{\partial x_j}$$

$$\mathcal{F} := \sum_{i,j}\frac{1}{\sqrt{|det(g)|}}\frac{\partial}{\partial x_i}F^{ij}\sqrt{|det(g)|}\frac{\partial}{\partial x_j}$$

Remark 9. The first operator is the Beltrami-Laplace operator of the metric g; another way to write it down is

$$\Delta_g = -\sum_{i,j} g^{ij}\nabla_i\nabla_j,$$

where ∇ is the Levi-Cìvita connection of g. The second operator is a natural quantization of the function $\sum_{i,j} F^{ij} p_i p_j$ and another way to write it down is

$$\mathcal{F}_g = \sum_{i,j}\nabla_i F^{ij}\nabla_j.$$

In particular, both operators do not depend on the choice of the coordinate system.

Remark 10. The symbols of Δ_g and of $\mathcal{F}_g$ are $-2H := -2\sum_{i,j} g^{ij} p_i p_j$ and $\sum_{i,j} F^{ij} p_i p_j$, respectively.

Theorem 5. *Let $F = \sum_{i,j} F^{ij} p_i p_j + V(x,y)$ be a quadratic integral of the natural Hamiltonian system $\frac{1}{2}\sum_{i,j} g^{ij} p_i p_j + U(x,y)$ on T^*M^2. Then, the operators*

$$\mathcal{H} := \Delta_g - 2U$$

and

$$\mathcal{F} := \mathcal{F}_g + V$$

commute: $\mathcal{H} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{H}$.

Remark 11. The Riemannian analog of Theorem 5 follows from [9, 23, 29, 30].

Proof of Theorem 5. It is sufficient to check the statement at almost every point, i.e., for the metrics and the integrals from Theorem 4. Direct calculations shows that in this case the operators Δ_g and $\mathcal{F}_g$ are as in the following table:

	Liouville case	*Complex-Liouville case*	*Jordan-block case*
Δ_g	$\frac{-1}{X(x)-Y(y)}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)$	$\frac{-4}{\Im(h)}\frac{\partial^2}{\partial x \partial y}$	$\frac{-4}{1+x_1 Y'(y)}\frac{\partial^2}{\partial x \partial y}$
$\mathcal{F}_g$	$\frac{1}{X(x)-Y(y)}\left(X(x)\frac{\partial^2}{\partial y^2} - Y(y)\frac{\partial^2}{\partial x^2}\right)$	$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2\frac{\Re(h)}{\Im(h)}\frac{\partial^2}{\partial x \partial y}$	$\frac{\partial^2}{\partial x^2} - 2\frac{Y(y)}{1+xY'(y)}\frac{\partial^2}{\partial x \partial y}$

where h is a holomorphic function of $z = x + i \cdot y$.

To prove that $\mathcal{H} = \Delta_g - 2U$ and $\mathcal{F} = \mathcal{F}_g + V$ commute, we first observe that in the Liouville and Jordan-block cases:

$$\mathcal{F}_g + V = \frac{\partial^2}{\partial x^2} + f \cdot (\Delta_g - 2U) + f_1,$$

where $f = X(x)$, $f_1 = \hat{X}(x)$ for the Liouville case, and $f = \frac{Y(y)}{2}$ and $f_1 = Y_1(y)$ for the Jordan block case.

Similarly, in the complex Liouville case, we have

$$\mathcal{F}_g + V = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + f \cdot (\Delta_g - 2U) + f_1$$

where $f = -\frac{\Re(h)}{2}$, $f_1 = \frac{\Re(h_1)}{2}$

The Laplace-Beltrami operator Δ_g in all the cases is of the form $\Delta_g = \lambda^{-1}\Delta_{g_0}$, where Δ_{g_0} is the Laplace-Beltrami operator of the flat metric g_0 (more specifically, g_0 is $dx^2 - dy^2$ in the Liouville case, and $2dxdy$ in the complex Liouville and Jordan-block cases). Using the fact that Δ_{g_0} commutes with $\frac{\partial}{\partial x}$, it is straightforward to verify the following commutator formula:

$$[\Delta_g - 2U, \frac{\partial^2}{\partial x^2}] = \left(\frac{\lambda_{xx}}{\lambda} + 2\frac{\lambda_x}{\lambda}\frac{\partial}{\partial x}\right) \circ (\Delta_g - 2U) + 2\frac{(\lambda U)_{xx}}{\lambda} + 4\frac{(\lambda U)_x}{\lambda}\frac{\partial}{\partial x}$$

(here we use standard notation for the commutator of two linear operators $[\mathcal{A}, \mathcal{B}] = \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$).

The two following formulas are standard:

$$[\Delta_g - 2U, f \cdot (\Delta_g - 2U)] = \left(\Delta_g f - 2\,\mathrm{grad}_g f\right) \circ (\Delta_g - 2U)$$

and

$$[\Delta_g - 2U, f_1] = \Delta_g f_1 - 2\,\mathrm{grad}_g f_1,$$

where the vector field $\mathrm{grad}_g f$ is viewed as a first order differential operator, i.e., $\mathrm{grad}_g f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$.

Thus, in the Liouville and Jordan-block cases, we have:

$$[\mathcal{H}, \mathcal{F}] = [\Delta_g - 2U, \mathcal{F}_g + V] = \left(\frac{\lambda_{xx}}{\lambda} + 2\frac{\lambda_x}{\lambda}\frac{\partial}{\partial x} + \Delta_g f - 2\,\mathrm{grad}_g f\right) \circ (\Delta_g - 2U) +$$
$$+ 2\frac{(\lambda U)_{xx}}{\lambda} + 4\frac{(\lambda U)_x}{\lambda}\frac{\partial}{\partial x} + \Delta_g f_1 - 2\,\mathrm{grad}_g f_1$$

Hence, the commutativity condition $[\mathcal{H},\mathcal{F}] = \mathcal{H}\circ\mathcal{F} - \mathcal{H}\circ\mathcal{F} = 0$ splits into four simple equations (here we use the fact that $\Delta_g = \lambda^{-1}\Delta_{g_0}$ and $\mathrm{grad}_g = \lambda^{-1}\mathrm{grad}_{g_0}$):

$$
\begin{array}{ll}
\text{(i)} & \lambda_x \frac{\partial}{\partial x} - \mathrm{grad}_{g_0} f = 0 \\
\text{(ii)} & \lambda_{xx} + \Delta_{g_0} f = 0 \\
\text{(iii)} & 2(\lambda U)_x \frac{\partial}{\partial x} - \mathrm{grad}_{g_0} f_1 = 0 \\
\text{(iv)} & 2(\lambda U)_{xx} + \Delta_{g_0} f_1 = 0
\end{array}
$$

Each of these equations has natural meaning. Indeed, (i) and (ii) mean that the operators Δ_g and $\mathcal{F}_g$ commute (without potentials), (iii) and (iv) give the "new" commutativity conditions involving the potentials. The first and third equations are equivalent to the commutativity of classical integrals, whereas the second and the fourth keep additional "quantum" information. It is interesting to notice that the quantum conditions (ii) and (iv) can be obtained from the classical ones (i) and (iii) by "differentiating" so that in our particular case the quantum integrability in dimension 2 turns out to be a corollary of the classical one:

$$
\begin{array}{l}
\lambda_{xx} + \Delta_{g_0} f = \mathrm{div}(\lambda_x \frac{\partial}{\partial x} - \mathrm{grad}_{g_0} f) \\
2(\lambda U)_{xx} + \Delta_{g_0} f_1 = \mathrm{div}(2(\lambda U)_x \frac{\partial}{\partial x} - \mathrm{grad}_{g_0} f_1)
\end{array}
$$

However, each of the above four conditions can be verified directly. Taking into account the following explicit formulas:

$$
\begin{array}{lll}
\text{Liouville case :} & & \\
\Delta_{g_0} = -\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}, & \mathrm{grad}_{g_0} f = f_x \frac{\partial}{\partial x} - f_y \frac{\partial}{\partial y}, & \lambda = X(x) - Y(y) \\
\text{Jordan-block case :} & & \\
\Delta_{g_0} = -2\frac{\partial^2}{\partial x \partial y}, & \mathrm{grad}_{g_0} f = f_y \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial y}, & \lambda = 1/2(1 + xY'(y))
\end{array}
$$

we see that equations(i)–(iv) become:

Liouville case:

$$
\begin{array}{l}
(\lambda_x - f_x)\frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y} = 0 \\
\lambda_{xx} - f_{xx} + f_{yy} = 0 \\
(2(\lambda U)_x - (f_1)_x)\frac{\partial}{\partial x} + (f_1)_y \frac{\partial}{\partial y} = 0 \\
2(\lambda U)_{xx} - (f_1)_{xx} + (f_1)_{yy} = 0
\end{array}
$$

Jordan-block case:

$$
\begin{array}{l}
(\lambda_x - f_y)\frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y} = 0 \\
\lambda_{xx} - 2f_{xy} = 0 \\
(2(\lambda U)_x - (f_1)_y)\frac{\partial}{\partial x} - (f_1)_x \frac{\partial}{\partial y} = 0 \\
2(\lambda U)_{xx} - 2(f_1)_{xy} = 0
\end{array}
$$

and obviously hold for λ, f and f_1 indicated above.

The complex Liouville case is absolutely similar, the only difference is the additional term $\frac{\partial^2}{\partial y^2}$, which leads to the following system of relations:

Complex Liouville case:

$$
\begin{array}{l}
(\lambda_x - f_y)\frac{\partial}{\partial x} + (-\lambda_y - f_x)\frac{\partial}{\partial y} = 0 \\
\lambda_{xx} - \lambda_{yy} - 2f_{xy} = 0 \\
(2(\lambda U)_x - (f_1)_y)\frac{\partial}{\partial x} + (-2(\lambda U)_y - (f_1)_x)\frac{\partial}{\partial y} = 0 \\
2(\lambda U)_{xx} - 2(\lambda U)_{yy} - 2(f_1)_{xy} = 0
\end{array}
$$

each of which obviously holds for $f = -\frac{\Re(h)}{2}$, $f_1 = -\Re(h_1)$, $\lambda = \frac{\Im(h)}{2}$, $2\lambda U = \Im(h_2)$. □

3 Proof of Theorem 1

3.1 Admissible coordinate systems and Birkhoff-Kolokoltsov forms

Let g be a pseudo-Riemannian metric on M^2 of signature $(+,-)$. Consider (and fix) two vector fields V_1, V_2 on M^2 such that

- $g(V_1, V_1) = g(V_2, V_2) = 0$ and

- $g(V_1, V_2) > 0$.

Such vector fields always exist locally, (and since our result is local, this is sufficient for our proof). For possible further use, let us note that such vector fields always exist on a finite (at most, 4-sheet-) cover of M^2.

We will say that a local coordinate system (x, y) is **admissible**, if the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are proportional to V_1, V_2 with positive coefficient of proportionality:

$$\frac{\partial}{\partial x} = \lambda_1(x,y)V_1(x,y), \quad \frac{\partial}{\partial y} = \lambda_2(x,y)V_2(x,y), \quad \text{where } \lambda_i > 0.$$

Obviously,

- admissible coordinates exist in a sufficiently small neighbourhood of every point,
- the metric g in admissible coordinates has the form

$$ds^2 = f(x,y)dxdy, \quad \text{where } f > 0, \tag{21}$$

- two admissible coordinate systems in one neighbourhood are connected by

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} x_{new}(x_{old}) \\ y_{new}(y_{old}) \end{pmatrix}, \quad \text{where } \frac{dx_{new}}{dx_{old}} > 0, \ \frac{dy_{new}}{dy_{old}} > 0. \tag{22}$$

Lemma 1. *Let (x, y) be an admissible coordinate system for g. Let F given by (1) be an integral for g. Then,*

$$B_1 := \frac{1}{\sqrt{|a(x,y)|}}dx, \quad \left(\text{respectively}, B_2 := \frac{1}{\sqrt{|c(x,y)|}}dy\right)$$

is a 1-form, which is defined at points such that $a \neq 0$ (respectively, $c \neq 0$). Moreover, the coefficient a (respectively, c) depends only on x (respectively, y), which in particular implies that the forms B_1, B_2 are closed.

Remark 12. The forms B_1, B_2 are not the direct analog of the "Birkhoff" 2-form introduced by Kolokoltsov in [22]. In a certain sense, they are a real analog of the square root of the Birkhoff form.

Proof of Lemma 1. The first part of the statement, namely that

$$\frac{1}{\sqrt{|a(x,y)|}}dx, \quad \left(\text{respectively}, \frac{1}{\sqrt{|c(x,y)|}}dy\right)$$

transforms as a 1-form under admissible coordinate changes is evident: indeed, after the coordinate change (22), the momenta transform as follows: $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$, $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$. Then, the integral F in the new coordinates has the form

$$\underbrace{\left(\frac{dx_{new}}{dx_{old}}\right)^2 a}_{a_{new}} p_{x_{new}}^2 + \underbrace{\frac{dx_{new}}{dx_{old}}\frac{dy_{new}}{dy_{old}} b}_{b_{new}} p_{x_{new}} p_{y_{new}} + \underbrace{\left(\frac{dy_{new}}{d_{y_{old}}}\right)^2 c}_{c_{new}} p_{y_{new}}^2.$$

Then, the formal expression $\frac{1}{\sqrt{|a|}}dx_{old}$ (respectively, $\frac{1}{\sqrt{|c|}}dy_{old}$) transforms into

$$\frac{1}{\sqrt{|a|}}\frac{dx_{old}}{dx_{new}}dx_{new} \qquad \left(\text{respectively}, \frac{1}{\sqrt{|c|}}\frac{dy_{old}}{dy_{new}}dy_{new}\right),$$

which is precisely the transformation law of 1-forms.

Let us prove that the coefficient a (respectively, c) depends only on x (respectively, y), which in particular implies that the forms B_1, B_2 are closed. If g is given by (21), its Hamiltonian is

$$H = \frac{2p_xp_y}{f},$$

and the condition $\{H, F\} = 0$ reads

$$\begin{aligned} 0 &= \left\{\frac{2p_xp_y}{f}, ap_x^2 + bp_xp_y + cp_y^2\right\} \\ &= \frac{2}{f^2}\left(p_x^3(fa_y) + p_x^2p_y(fa_x + fb_y + 2f_xa + f_yb) + p_yp_x^2(fb_x + fc_y + f_xb + 2f_yc) + p_y^3(c_xf)\right), \end{aligned}$$

i.e., is equivalent to the following system of PDE:

$$\left\{\begin{array}{rcl} a_y &=& 0, \\ fa_x + fb_y + 2f_xa + f_yb &=& 0, \\ fb_x + fc_y + f_xb + 2f_yc &=& 0, \\ c_x &=& 0. \end{array}\right. \tag{23}$$

Thus, $a = a(x)$, $c = c(y)$, which is equivalent to state that $B_1 := \frac{1}{\sqrt{|a|}}dx$ and $B_2 := \frac{1}{\sqrt{|c|}}dy$ are closed forms (assuming $a \neq 0$ and $c \neq 0$). □

Remark 13. For further use let us formulate one more consequence of equations (23): if $a \equiv c \equiv 0$ in a neighbourhood of a point, then $bf = \text{const}$, implying $F \equiv \text{const} \cdot H$ in the neighbourhood.

Assume $a \neq 0$ (respectively, $c \neq 0$) at a point p_0. For every p_1 in a small neighbourhood U of p_0 consider

$$x_{new} := \int_{\gamma : [0,1] \to U} B_1, \quad \left(\text{respectively, } y_{new} := \int_{\gamma : [0,1] \to U} B_2\right), \tag{24}$$

with $\gamma(0) = p_0, \gamma(1) = p_1$.

Locally, in the admissible coordinates, the functions x_{new} and y_{new} are given by

$$x_{new}(x) = \int_{x_0}^{x} \frac{1}{\sqrt{|a(t)|}}\, dt, \qquad y_{new}(y) = \int_{y_0}^{y} \frac{1}{\sqrt{|c(t)|}}\, dt. \tag{25}$$

The coordinates (x_{new}, y_{old}), $\big((x_{old}, y_{new})$, (x_{new}, y_{new}), respectively$\big)$ are admissible. In these coordinates the forms B_1, B_2 are given by dx_{new}, dy_{new} implying that $a = c = \pm 1$ (more precisely: $a_{new} = \text{sign}(a_{old})$, $c_{new} = \text{sign}(c_{old})$).

3.2 Proof of Theorem 1

We assume that g on M^2 of signature (+,–) admits a nontrivial quadratic integral F given by (1). Consider the $(1,1)$-tensor G given by (2). In a neighbourhood of almost every point, the Jordan normal form of this $(1,1)$-tensor is one of the following:

Case 1 $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

Case 2 $\begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

Case 3 $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

Moreover, in view of Remark 13, there exists a neighbourhood of almost every point such that $\lambda \neq \mu$ in case 1 and $\mu \neq 0$ in case 2. In the admissible coordinates, up to multiplication of F by -1, case 1 is equivalent to the condition $ac > 0$, case 2 is equivalent to the condition $ac < 0$ and, finally, case 3 is equivalent to the condition $ac = 0$.

We now consider all three cases.

3.2.1 Case 1: $ac > 0$.

Without loss of generality we assume $a > 0$, $c > 0$. Consider the coordinates (24). In these coordinates $a = 1$, $c = 1$ and equations (23) have the following simple form.

$$\begin{cases} (fb)_y + 2f_x &= 0\,, \\ (fb)_x + 2f_y &= 0\,. \end{cases} \tag{26}$$

This system can be solved. Indeed, it is equivalent to

$$\begin{cases} (fb+2f)_x + (fb+2f)_y &= 0\,, \\ (fb-2f)_x - (fb-2f)_y &= 0\,, \end{cases} \tag{27}$$

which after the (non-admissible) change of coordinates $x_{new} = x + y$, $y_{new} = x - y$, has the form

$$\begin{cases} (fb+2f)_x &= 0\,, \\ (fb-2f)_y &= 0\,, \end{cases} \tag{28}$$

implying $fb + 2f = Y(y)$, $fb - 2f = X(x)$. Thus,

$$f = \frac{Y(y) - X(x)}{4}\,, \quad b = 2\frac{X(x) + Y(y)}{Y(y) - X(x)}\,.$$

Finally, in the new coordinates, the metric and the integral have (up to a possible multiplication by a constant) the form

$$(X - Y)(dx^2 - dy^2)\,, \tag{29}$$

$$\frac{1}{2}\left(p_x^2 - \tfrac{X(x)+Y(y)}{X(x)-Y(y)}(p_x^2 - p_y^2) + p_y^2\right) = \frac{p_y^2 X(x) - p_x^2 Y(y)}{X(x) - Y(y)}\,. \tag{30}$$

3.2.2 Case 2: $ac < 0$.

Without loss of generality we can assume $a > 0$, $c < 0$. Consider the normal coordinates (24). In these coordinates $a = 1$, $c = -1$ and equations (23) have the following simple form.

$$\begin{cases} (fb)_y + 2f_x &= 0\,, \\ (fb)_x - 2f_y &= 0\,. \end{cases} \tag{31}$$

We see that these equations are the Cauchy-Riemann conditions for the complex-valued function $fb + 2if$. Thus, for an appropriate holomorphic function $h = h(x + iy)$ we have $fb = \Re(h)$, $2f = \Im(h)$.

Finally, in a certain coordinate system, the metric and the integral are (up to possible multiplication by constants)

$$\Im(h)dxdy \quad \text{and} \quad p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y \tag{32}$$

3.2.3 Case 3: $ac = 0$.

Without loss of generality we can assume $a > 0$, $c = 0$. Consider admissible coordinates x, y, such that x is the normal coordinate from (24). In these coordinates $a = 1$, $c = 0$, and the equations (23) have the following simple form.

$$\begin{cases} (fb)_y + 2f_x &= 0\,, \\ (fb)_x &= 0\,. \end{cases} \tag{33}$$

This system can be solved. Indeed, the second equation implies $fb = -Y(y)$. Substituting this in the first equation we obtain $Y' = 2f_x$ implying

$$f = \frac{x}{2}Y'(y) + \widehat{Y}(y) \quad \text{and} \quad b = -\frac{Y(y)}{\frac{x}{2}Y'(y) + \widehat{Y}(y)}\,.$$

Finally, the metric and the integral are

$$\left(\widehat{Y}(y) + \frac{x}{2}Y'(y)\right) dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{\widehat{Y}(y) + \frac{x}{2}Y'(y)} p_x p_y \,. \tag{34}$$

Moreover, by the change $y_{new} = \beta(y_{old})$, equations (34) will be simply transformed to:

$$\left(\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)\right) dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)} p_x p_y \,. \tag{35}$$

Thus, by putting $\beta(y) = \int_{y_0}^{y} \frac{1}{\widehat{Y}(t)} dt$, we can make the metric and the integral to be

$$\left(1 + \frac{x}{2}Y'(y)\right) dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{1 + \frac{x}{2}Y'(y)} p_x p_y \,.$$

Moreover, after the coordinate change $x_{new} = \frac{x_{old}}{2}$ and multiplication of the metric by $\frac{1}{2}$, the metric and the integral have the form from Theorem 1

$$\left(1 + xY'(y)\right) dxdy \quad \text{and} \quad p_x^2 - 2\frac{Y(y)}{1 + xY'(y)} p_x p_y \,. \tag{36}$$

Theorem 1 is proved.

Remark 14. Let us note that if $dY \neq 0$, then we can take Y as the coordinate y. Then, the metric and the integral (34) will have the form (see also Remark 4)

$$\left(\tilde{Y}(y) - \frac{x}{2}\right) dxdy \quad \text{and} \quad p_x^2 + \frac{y}{\tilde{Y}(y) - \frac{x}{2}} p_x p_y \,. \tag{37}$$

4 Conclusions

We have discussed integrable geodesic flows of pseudo-Riemannian metrics on 2-dimensional manifolds constructing (Theorem 1) local normal forms of such metrics. The normal forms are of three types: Liouville (the analogous of the Riemannian case), Complex-Liouville and Jordan-block. We have shown that these metrics, in analogy with the Riemannian case, admit geodesically equivalent metrics, can be used to construct a large family of natural systems admitting integrals quadratic in momenta, that these natural systems are integrable by quadratures and that the integrability of such systems can be generalized to the quantum setting. A natural further step in this field would be to understand what is the structure of the quadratic integral in the case in which the manifold is closed (the Riemannian case is done in [4, 8, 21, 22]).

Acknowledgement

The first author thanks the Russian Foundation for Basic Research (grants 05-01-00978) for partial financial support. The second author thanks Deutsche Forschungsgemeinschaft (Priority Program 1154 — Global Differential Geometry) for partial financial support and Loughborough and Cambridge Universities, and also Università di Roma "Tor Vergata" for their hospitality. The third author acknowledges the financial support from INFN, Sezione di Roma II.

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