Complete commutative subalgebras in polynomial Poisson algebras: a proof of the Mischenko–Fomenko conjecture

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Abstract

The Mishchenko-Fomenko conjecture says that for each real or complex finite-dimensional Lie algebra $\mathfrak{g}$ there exists a complete set of commuting polynomials on its dual space $\mathfrak{g}^*$. In terms of the theory of integrable Hamiltonian systems this means that the dual space $\mathfrak{g}^*$ endowed with the standard Lie-Poisson bracket admits polynomial integrable Hamiltonian systems. Recently this conjecture has been proved by S.T. Sadetov. Following his idea, we give an explicit geometric construction for commuting polynomials on $\mathfrak{g}^*$ and consider some examples.

1 Introduction and preliminaries

Consider a symplectic manifold (M^{2n}, ω) and a Hamiltonian system $\dot{x} = X_H(x)$ on it, where $H : M^{2n} \to \mathbb{R}$ is a smooth function called *Hamiltonian* and $X_H(x) = \omega^{-1}(dH(x))$ is the corresponding Hamiltonian vector field.

This system is called *completely integrable* if it admits n functionally independent integrals $f_1, \dots, f_n : M^{2n} \to \mathbb{R}^n$ which commute with respect to the Poisson bracket associated with the symplectic structure ω, i.e., $\{f_i, f_j\} = 0$, $i, j = 1, \dots, n$.

Equivalently one can say that this system admit a complete commutative subalgebra $\mathcal{A}$ of integrals in the Poisson algebra $C^\infty(M^{2n})$ of smooth functions on M. Completeness means that at a generic point $x \in M^{2n}$, the subspace in T^*M generated by the differentials $df(x)$, $f \in \mathcal{A}$ is maximal isotropic.

The same definition makes sense if, instead of a symplectic manifold, we consider a Poisson manifold $(M, \{,\})$ where the Poisson bracket $\{,\}$ is not necessarily non-degenerate.

One of the most intriguing questions in the theory of integrable systems can be formulated as follows: does a given symplectic (Poisson) manifold M admit an integrable system with nice properties?

Notice that the necessity of "nice properties" is motivated by the fact that any symplectic (Poisson) manifold admits a smooth integrable system which can be constructed by using some kind of "partition of unity" idea

[11]. The behavior of such a system, however, has no relation to the geometry of the underlying manifold and therefore is not of interest at all.

The additional assumptions that make the above question non-trivial and interesting can be rather various. Briefly, we mention three types of integrable systems for which the existence problem is extremely interesting and important:

1) toric (or almost toric) integrable systems [8, 1, 26, 31];

2) integrable systems with non-degenerate singularities [9], [20], [21], [5];

3) integrable geodesic flows on compact manifolds [13], [7], [22], [4].

In the algebraic case, the existence problem seems to be interesting even without any additional assumptions: given an algebraic symplectic (Poisson) manifold X, does it admit a polynomial (rational) integrable system? In the present paper, we discuss this problem in the case when X is a dual space of a finite-dimensional Lie algebra endowed with the standard linear Lie-Poisson bracket.

We start with recalling basic definitions. Consider a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ and its dual space $\mathfrak{g}^*$ endowed with the standard Poisson-Lie structure which is defined as follows. Let $f, g : \mathfrak{g}^* \to \mathbb{R}$ be arbitrary smooth functions. Their differentials at a point $x \in \mathfrak{g}^*$ can be treated as elements of the Lie algebra $\mathfrak{g}$. Then the Lie-Poisson bracket of f and g is defined by:

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle. \tag{1}$$

If instead of smooth functions we restrict ourselves with polynomials on $\mathfrak{g}^*$, then the same operation can be introduced in the following equivalent way. The Poisson-Lie bracket on the space of polynomials is defined to be a bilinear skew-symmetric operation satisfying two properties:

1) $\{fg, h\} = f\{g, h\} + g\{f, h\}$ (Leibniz rule);

2) if $f, g \in \mathfrak{g}$ are linear polynomials on $\mathfrak{g}^*$ then the Poisson-Lie bracket coincides with the usual commutator in $\mathfrak{g}$, i.e.,

$$\{f, g\} = [f, g].$$

The space of polynomials $\mathbb{R}[\mathfrak{g}]$ with such an operation is called the Poisson algebra (associated with $\mathfrak{g}$) and is denoted by $S(\mathfrak{g})$.

The Poisson-Lie bracket is naturally extended to the space of rational functions $\mathbb{R}(\mathfrak{g}) = \mathrm{Frac}(S(\mathfrak{g}))$, and (which is very important for our considerations) all the definitions make sense over arbitrary field $\mathbb{K}$ of zero characteristic.

To each finite-dimensional Lie algebra (over a field $\mathbb{K}$) one can assign two integer numbers: its dimension $\dim \mathfrak{g}$ and index $\mathrm{ind}\, \mathfrak{g}$. The latter is the corank of the skew-symmetric form $\Phi_x : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ for a generic element $x \in \mathfrak{g}^*$ where

$$\Phi_x(\xi, \eta) = \langle x, [\xi, \eta]\rangle.$$

Definition 1 *A commutative set of algebraically independent polynomials*

$$f_1, \ldots, f_k \in S(\mathfrak{g})$$

is called complete, if $k = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g})$.

A commutative subalgebra $\mathcal{A} \subset S(\mathfrak{g})$ is called complete if $\mathrm{tr.deg}\,\mathcal{A} = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g})$.

The completeness condition means that, at a generic point $x \in \mathfrak{g}^*$, the subspace in $\mathfrak{g}$ generated by the differentials $df_1(x), \dots, df_k(x)$ is maximal isotropic with respect to the Lie-Poisson bracket at x, i.e., in the sense of the skew-symmetric form Φ_x. In particular, the maximal possible number of commuting independent polynomials in $S(\mathfrak{g})$ cannot exceed $\frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\, \mathfrak{g})$.

Conjecture 1 (Mishchenko-Fomenko [18]) *Let $\mathfrak{g}$ be a real or complex finite-dimensional Lie algebra. Then on $\mathfrak{g}^*$ there exists a complete commutative set of polynomials.*

In more algebraic terms this means that each Poisson algebra $S(\mathfrak{g})$ admits a complete commutative subalgebra $\mathcal{A}$.

This conjecture comes from the theory of integrable Hamiltonian systems and can be reformulated as follows: on the dual space $\mathfrak{g}^*$ of every finite-dimensional Lie algebra $\mathfrak{g}$ there exist integrable Hamiltonian systems with polynomial integrals.

In 1978 A. Mishchenko and A. Fomenko [16] proved this conjecture for semisimple Lie algebras. Since then complete commutative sets have been constructed for many other classes of Lie algebras (see [10], [2], [29], [28]). Recently S. Sadetov [25] has proved this conjecture in the general case by using one nice algebraic construction that reduces the problem either to the semisimple case, or to an algebra of smaller dimension.

Theorem 1 (Sadetov, 2003) *The Mishchenko-Fomenko conjecture holds for an arbitrary finite-dimensional Lie algebra over a field of zero characteristic.*

It is remarkable fact that working over an arbitrary field surprisingly simplifies the proof. The main construction is based on the induction argument. On each step we reduce the dimension of the Lie algebra in question, but we have to pay for this by extending the field. However, this price is not very high since all the statements and definitions admit purely algebraic formulations so that the field do not play any essential role.

The purpose of this paper is to present Sadetov's construction in more explicit terms of Poisson geometry allowing one to work effectively with specific Lie algebras. The approach suggested by S.Sadetov is, in fact, purely algebraic. In our opinion, however, behind his construction one can see important geometrical ideas which we would like to emphasize in the present paper rather that to give another rigorous proof. We also study several natural examples of Lie algebras and describe explicitly the related complete commutative subalgebras some of which are quite remarkable.

The proof which we are going to present is actually based on a modification of two well-known constructions: the "argument shift" method suggested by A. Mishchenko and A. Fomenko and the so-called "chain of subalgebras" method which was used by many authors for different purposes (see, in particular, Gelfand-Zetlin [12], Vergne [30], Thimm [28], Trofimov [29]). We start with recalling these constructions.

2 "Chain of subalgebras" method

In this section, by $\mathfrak{g}$ we mean a real or complex Lie algebra. However almost all constructions make sense for any field $\mathbb{K}$ of zero characteristic.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Suppose that we can construct a complete commutative subalgebra $\mathcal{A}$ in $S(\mathfrak{h})$. Since $S(\mathfrak{h}) \subset S(\mathfrak{g})$, we can try to extend $\mathcal{A}$ up to a complete commutative subalgebra in $S(\mathfrak{g})$. To this end we need to find additional polynomials $f_1, \dots, f_s$ which commute with $S(\mathfrak{h})$ and between themselves. As good candidates we can use, for examples, the invariants of the coadjoint representation of $\mathfrak{g}$ or, which is the same, the polynomials from the center $Z(S(\mathfrak{g}))$ of $S(\mathfrak{g})$. Sometimes these polynomials are sufficient to satisfy the completeness condition.

Repeating this idea for a chain of subalgebras

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$$

we can always construct a "big" set of commuting polynomials:

$$Z_0 \cup Z_1 \cup \dots \cup Z_{n-1} \cup Z_n,$$

where $Z_i = Z(S(\mathfrak{g}_i))$.

For many important cases this allows us to construct a complete commutative subalgebra in $S(\mathfrak{g})$. For example, it is so for the chains (see [28])

$$gl(1,\mathbb{R}) \subset gl(2,\mathbb{R}) \subset \dots \subset gl(n-1,\mathbb{R}) \subset gl(n,\mathbb{R}),$$
$$so(1,\mathbb{R}) \subset so(2,\mathbb{R}) \subset \dots \subset so(n-1,\mathbb{R}) \subset so(n,\mathbb{R}),$$

and also for codimension one filtrations in nilpotent (see [30]) and solvable algebraic Lie algebras (in the latter case instead of polynomials one has to consider rational functions, but after some modification using semi-invariants instead of invariants one still can solve the problem without leaving the space of polynomials).

However in the general case an appropriate chain of subalgebras does not always exist, and this method does not work directly.

Let us look at the problem with more attention. To understand the situation better, let us first consider the following "linear" version of our problem. Take a vector space V endowed with a skew-symmetric bilinear form ϕ (possibly, degenerate!). Let $U_1 \subset V$ be a subspace, and $A_1 \subset U_1$ be a maximal isotropic subspace in U_1. The problem is to extend A_1 up to a maximal isotropic subspace $A \subset V$. One of possible solutions is the following. Consider the skew-orthogonal "complement" of U_1 in V, i.e., subspace

$$U_2 = U_1^{\phi} = \{v \in V \mid \phi(u,v) = 0 \text{ for all } u \in U_1\}.$$

Let $A_2 \subset U_2$ be a maximal isotropic subspace in U_2. Then $A = A_1 + A_2$ is maximal isotropic in V. This is a simple fact from linear symplectic geometry.

We now consider a "non-linear" version of this statement. Consider a Poisson manifold (X, ϕ) and a (Poisson) subalgebra $\mathcal{F} \subset C^\infty(X)$. A commutative subalgebra $\mathcal{A} \subset \mathcal{F}$ is called complete in $\mathcal{F}$ if at a generic point $x \in M$ the following condition holds. Consider the subspaces $d\mathcal{A}(x)$

and $d\mathcal{F}(x)$ in T_x^*X generated by the differentials of functions f from $\mathcal{A}$ and $\mathcal{F}$ respectively. It is clear that $d\mathcal{A}(x)$ is an isotropic subspace in $d\mathcal{F}(x)$ with respect to the Poisson structure ϕ.

Definition 2 *A commutative subalgebra $\mathcal{A} \subset \mathcal{F}$ is called complete in $\mathcal{F}$ if $d\mathcal{A}(x)$ is maximal isotropic in $d\mathcal{F}(x)$ at a generic point $x \in X$.*

Now consider two (Poisson) subalgebras $\mathcal{F}_1, \mathcal{F}_2 \subset C^\infty(X)$ such that $\{\mathcal{F}_1, \mathcal{F}_2\} = 0$. Let $\mathcal{A}_1 \subset \mathcal{F}_1$, $\mathcal{A}_2 \subset \mathcal{F}_2$ be complete commutative subalgebras in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. The following proposition is just a reformulation of the "linear" statement.

Proposition 1 *Suppose $d\mathcal{F}_2(x) = d\mathcal{F}_1(x)^\phi = \{\xi \in T_x^*X \mid \phi(\xi, df(x)) = 0$ for any $f \in \mathcal{F}_1\}$ at a generic point $x \in X$. Then $\mathcal{A}_1 + \mathcal{A}_2$ is a complete commutative subalgebra in $C^\infty(X)$.*

Here by *generic* we mean "from open everywhere dense subset" without specifying the nature of such a subset, and $\mathcal{A}_1 + \mathcal{A}_2$ denotes the least Poisson subalgebra in $C^\infty(X)$ which contains both $\mathcal{A}_1$ and $\mathcal{A}_2$.

Remark 1 The condition $d\mathcal{F}_2(x) = d\mathcal{F}_1(x)^\phi$ can be replaced by the following assumption: $d\mathcal{F}_2(x) + d\mathcal{F}_1(x)$ is coisotropic in $T_x^*(X)$, which is slightly weaker.

This simple idea can now be applied to our problem. Having a complete commutative subalgebra $\mathcal{A} \subset S(\mathfrak{h})$, we need to extend it up to a complete commutative subalgebra in $S(\mathfrak{g})$. Following the above construction, we should consider the maximal subalgebra in $S(\mathfrak{g})$ all of whose elements commute with $S(\mathfrak{h})$. Since $S(\mathfrak{h})$ is generated by $\mathfrak{h}$, this subalgebra is:

$$\mathrm{Ann}(\mathfrak{h}) = \{f \in S(\mathfrak{g}) \mid \{f, \eta\} = 0, \quad \forall\, \eta \in \mathfrak{h}\}.$$

It is easy to see that $\mathrm{Ann}(\mathfrak{h})$ consists exactly of invariant polynomials with respect to the coadjoint action of H on $\mathfrak{g}^*$, where $H \subset G$ is the Lie subgroup corresponding to $\mathfrak{h}$. To apply Proposition 1 we have to assume that this representation admits sufficiently many polynomial invariants. More precisely, this means that elements of $\mathrm{Ann}(\mathfrak{h})$ distinguish generic orbits, i.e.,

$$\mathrm{tr.\,deg}\,\mathrm{Ann}(\mathfrak{h}) = \mathrm{codim}\,\mathcal{O}_H(x), \tag{2}$$

where $\mathcal{O}_H(x) \subset \mathfrak{g}$ is a generic Ad_H^*-orbit.

Notice that this condition means exactly that

$$d\,\mathrm{Ann}(\mathfrak{h})(x) = \mathfrak{h}^{\Phi_x} = \{\xi \in \mathfrak{g} \mid \langle x, [\xi, \eta]\rangle = 0\ \forall \eta \in \mathfrak{h}\}$$

at a generic point $x \in \mathfrak{g}^*$ and we can reformulate Proposition 1 as follows.

Proposition 2 *Let $\mathfrak{h}$ and $\mathrm{Ann}(\mathfrak{h})$ both admit complete commutative subalgebras of polynomials $\mathcal{A}_1 \subset S(\mathfrak{h})$ and $\mathcal{A}_2 \subset \mathrm{Ann}(\mathfrak{h})$ respectively. If (2) holds, then $\mathcal{A}_1 + \mathcal{A}_2$ is a complete commutative subalgebra in $S(\mathfrak{g})$.*

Remark 2 If we work over an arbitrary field $\mathbb{K}$ of zero characteristic, then condition (2) is not so convenient and can be replaced by one of the two following assumptions which do not involve any Lie groups:

1) $d\,\mathrm{Ann}(\mathfrak{h})(x) + \mathfrak{h}$ is a coisotropic subspace in $\mathfrak{g}$ w.r.t. Φ_x for generic $x \in \mathfrak{g}^*$.

2) $\operatorname{tr.deg} \operatorname{Ann}(\mathfrak{h}) = \operatorname{codim} \operatorname{ad}^*_{\mathfrak{h}} x$ for generic $x \in \mathfrak{g}^*$ (in the classical case where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, this subspace $\operatorname{ad}^*_{\mathfrak{h}} x \subset \mathfrak{g}^*$ is just the tangent space for the orbit $\mathcal{O}_H(x)$ at x).

Thus, to construct a complete commutative subalgebra in $S(\mathfrak{g})$, it suffices to find complete commutative subalgebras in $S(\mathfrak{h})$ and $\operatorname{Ann}(\mathfrak{h})$. Usually the dimension of $\mathfrak{h}$ and the transcendence degree of $\operatorname{Ann}(\mathfrak{h})$ are both smaller than $\dim \mathfrak{g}$, and we may hope that the problem of constructing complete commutative subalgebras in $S(\mathfrak{h})$ and $\operatorname{Ann}(\mathfrak{h})$ will be simpler than that in $S(\mathfrak{g})$. The difficulty, however, is that $\operatorname{Ann}(\mathfrak{h})$ may have a rather complicated algebraic structure.

It appears, nevertheless, that each non-semisimple Lie algebra always admits an ideal $\mathfrak{h} \subset \mathfrak{g}$ such that $\operatorname{Ann}(\mathfrak{h})$ has a very nice structure. Roughly speaking, $\operatorname{Ann}(\mathfrak{h})$ can be treated as a symmetric algebra $S(L)$ of a certain finite-dimensional Lie algebra L but perhaps over a new field $\mathbb{K}$. After this, according to Proposition 2 our problem is reduced to the same problem for smaller algebras $\mathfrak{h}$ and L, which allows us to use the induction argument.

3 Argument shift method

The argument shift method was suggested by A.T. Fomenko and A.S. Mishchenko in [16] as a generalization of S.V. Manakov's construction [15].

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ be its dual space. Consider the ring of invariants of the coadjoint representation $\operatorname{Ad}^* : G \to GL(\mathfrak{g}^*)$:

$$I_{\operatorname{Ad}^*}(G) = \{f : \mathfrak{g}^* \to \mathbb{R} \mid f(l) = f(\operatorname{Ad}^*_g l) \quad \text{for any } g \in G\}$$

Generally speaking, the Ad^*-invariants are not necessarily polynomials. But locally in a neighdorhood of a regular element $x \in \mathfrak{g}^*$ we always can find $k = \operatorname{ind} \mathfrak{g}$ functionally independent smooth invariants.

For a fixed regular element $a \in \mathfrak{g}^*$, consider the family of functions

$$\mathcal{F}_a = \{f_\lambda(x) = f(x + \lambda a)\}_{f \in I_{\operatorname{Ad}^*}(G), \lambda \in \mathbb{R}}.$$

It turns out that this family is commutative with respect to the Lie-Poisson structure. As we already noticed, the commuting functions so obtained are not necessarily polynomials. However, this trouble can be avoided by replacing the functions $f(x + \lambda a)$ with the homogeneous polynomials $f_k(x)$ obtained by Taylor expansion of $f(x)$ at the point $a \in \mathfrak{g}^*$:

$$f(a + \lambda x) = f(a) + \lambda f_1(x) + \lambda^2 f_2(x) + \dots$$

As a result, we shall obtain a commutative subset $\{f_k\}_{f \in I_{\operatorname{Ad}^*}(G)}$ in $S(\mathfrak{g})$ which we shall still denote by $\mathcal{F}_a$.

Theorem 2 (Mishchenko, Fomenko [16]) *If $\mathfrak{g}$ is semisimple and $a \in \mathfrak{g}^*$ is a regular element, then the commutative set $\mathcal{F}_a$ is complete.*

It is well known that the argument shift method is closely related to compatible Poisson brackets and bi-Hamiltonian systems. Indeed, on $\mathfrak{g}^*$ there are two natural compatible Poisson brackets. The first one is the standard Poisson-Lie bracket (1), the second is given by

$$\{f, g\}_a(x) = a([df(x), dg(x)]),$$

where $a \in \mathfrak{g}^*$ is a fixed element.

The compatibility condition is straightforward and the bi-hamiltonian approach (see [14]) leads us immediately to Hamiltonian systems whose first integrals are Casimir functions of linear combinations $\{,\} + \lambda\{,\}_a$, which coincide exactly with the functions from $\mathcal{F}_a$.

The bi-hamiltonian approach can be applied for an arbitrary Lie algebra, not necessarily semi-simple, and in fact, the family $\mathcal{F}_a$ turns out to be complete for many other classes of Lie algebras. More precisely, the following criterion holds.

Consider the set of singular elements in $\mathfrak{g}^*$:

$$\mathrm{Sing} = \{l \in \mathfrak{g}^* \mid \dim \mathrm{St}_{\,\mathrm{ad}^*}(l) > \mathrm{ind}\, \mathfrak{g}\},$$

where $\mathrm{St}_{\,\mathrm{ad}^*}(l) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi\, l = 0\}$ is the stationary subalgebra of l in the sense of the coadjoint representation.

If $\mathfrak{g}$ is an algebra over $\mathbb{R}$, then Sing is taken in the complexification $(\mathfrak{g}^{\mathbb{C}})^*$.

Theorem 3 ([2]) *Let $a \in \mathfrak{g}^*$ be a regular element. The commutative set $\mathcal{F}_a \subset S(\mathfrak{g})$ is complete if and only if* $\mathrm{codim}\, \mathrm{Sing} > 1$.

It is important to remark that in the semisimple case the argument shift method works for any field of zero characteristic. This follows from the fact that the completeness condition is preserved under extension of the field.

We now consider an example of a semisimple Lie algebra over a "non-standard" field to show how the argument shift methods works in a more complicated situation.

Consider a linear representation ρ of a complex Lie algebra $\mathfrak{g}$ on a linear space V.

Consider all rational mappings $\Psi : V \to \mathfrak{g}$ satisfying the following property: $\Psi(v) \in \mathrm{St}\,(v)$ where $\mathrm{St}\,(v) = \{\xi \in \mathfrak{g} \mid \rho(\xi)v = 0\}$ is the stationary subalgebra of v with respect to ρ.

In other terms, Ψ can be treated as a rational section of the stationary subalgebra fiber bundle over V (the fact that these subalgebras are of different dimensions is not important, over an Zariski open set this fiber bundle is smooth and locally trivial).

It is easy to see that the space $L = L(\mathfrak{g}, \rho, V)$ of such sections can be endowed with a Lie algebra structure. Indeed, we can just put by definition:

$$[\Psi_1, \Psi_2](v) = [\Psi_1(v), \Psi_2(v)] \in \mathrm{St}\,(v).$$

Over the original field this Lie algebra $L = L(\mathfrak{g}, \rho, V)$ is infinite dimensional. But, we can, obviously, consider it over the field $\mathbb{K} = \mathbb{C}(v_1, \dots, v_k)$ of rational functions on V. Then $L(\mathfrak{g}, \rho, V)$ has a finite dimension and, moreover, $\dim_{\mathbb{K}} L(\mathfrak{g}, \rho, V)$ is equal to the dimension (over $\mathbb{C}$) of a generic stationary subalgebra.

Assume that a generic stationary subalgebra $\mathrm{St}\,(v)$ is semisimple, then so is $L(\mathfrak{g}, \rho, V)$ over $\mathbb{K}$.

Let us construct a complete commutative set in $P(L(\mathfrak{g}, \rho, V))$ by using the argument shift method. First of all we notice that, as usual, L^* can be identified with L (and, consequently, ad with ad*) by using the form $\mathrm{Tr} : L \times L \to \mathbb{K}$:

$$(\mathrm{Tr}\,\Psi_1\Psi_2)(v) = \mathrm{Tr}_\rho\big(\Psi_1(v)\Psi_2(v)\big).$$

First of all, we need to describe the "(co)adjoint invariants" or, which is the same, the center of the corresponding Poisson algebra $S(L)$. Since $\mathrm{St}\,(v)$ can be considered as a semisimple Lie algebra in $gl(V)$, one can use the polynomial functions $F_k : L^*(\mathfrak{g}, \rho, V) = L(\mathfrak{g}, \rho, V) \to \mathbb{K}$ given by

$$F_k(\Psi) = \mathrm{Tr}_\rho\, \Psi(v)^k.$$

It is easy to see that $F_k \in S(L)$, $k = 1, 2, \dots$

Thus, the commuting polynomials in $S(L)$ constructed by the argument shift method can be written as follows:

$$F_{\lambda,k}(\Psi) = \mathrm{Tr}_\rho(\Psi(v) + \lambda\Psi_0(v))^k, \tag{3}$$

where $\Psi_0 : V \to \mathfrak{g}$ is a fixed rational section of the stationary subalgebra fiber bundle (in other words, $\Psi_0 \in L = L(\mathfrak{g}, \rho, V)$) satisfying one additional condition: for a generic $v \in V$, the corresponding element $\Psi_0(v)$ must be regular in $\mathrm{St}\,(v)$.

The completeness of the set of such polynomials (over $\mathbb{K}$) is evident. Indeed, the completeness condition for L is equivalent to the completeness condition for the functions $\mathrm{Tr}_\rho(X + \lambda A)^k$ defined on $\mathrm{St}\,(v)$ for generic $v \in V$ (here $X \in \mathrm{St}\,(v)$ is variable, $A \in \mathrm{St}\,(v)$ is fixed). But the last condition holds just because $\mathrm{St}\,(v)$ is a usual semisimple algebra over $\mathbb{C}$ (see Theorem 2).

4 Proof of the Mishchenko-Fomenko conjecture

Now we are ready to prove the Mishchenko-Fomenko conjecture. The following statement reduces the general situation to several separate cases.

Lemma 1 *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ of zero characteristic. Then one of the following statements holds:*

(i) $\mathfrak{g}$ has a commutative ideal $\mathfrak{h}$ which satisfies at least one of the two conditions: either $\dim\mathfrak{h} > 1$ *or* $[\mathfrak{h}, \mathfrak{g}] \neq 0$*;*

(ii) $\mathfrak{g}$ has an ideal $\mathfrak{h}$ isomorphic to the Heisenberg algebra $\mathfrak{h}_m$ and the center of $\mathfrak{g}$ coincides with the center of $\mathfrak{h}$;

(iii) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{K}$, where $\mathfrak{g}_0$ is semisimple;

(iv) $\mathfrak{g}$ is semisimple.

Proof. Consider the radical $\mathfrak{r}$ of $\mathfrak{g}$ (if $\mathfrak{r}$ is trivial, then $\mathfrak{g}$ is semisimple and we have (iv)). Take the chain or ideals:

$$\{0\} \subset \mathfrak{r}^{(k)} \subset \mathfrak{r}^{(k-1)} \subset \dots \subset \mathfrak{r}^{(1)} \subset \mathfrak{r}^{(0)} = \mathfrak{r}$$

where $\mathfrak{r}^{(l+1)} = [\mathfrak{r}^{(l)}, \mathfrak{r}^{(l)}]$. Obviously, $\mathfrak{r}^{(k)}$ is a commutative ideal. If $\dim\mathfrak{r}^{(k)} \neq 1$ or $\mathfrak{r}^{(k)}$ does not belong to the center $Z(\mathfrak{g})$ of $\mathfrak{g}$, then we get (i).

Assume that $\dim \mathfrak{r}^{(k)} = 1$ and $\mathfrak{r}^{(k)} \subset Z(\mathfrak{g})$. If the center itself is of dimension greater than 1, then we may take $Z(\mathfrak{g})$ as a commutative ideal satisfying (i).

If $\dim Z(\mathfrak{g}) = 1$, then $\mathfrak{r}^{(k)}$ coincides with $Z(\mathfrak{g})$ and there are two possibilities:

1) $\mathfrak{r}^{(k)} = \mathfrak{r}$ and then we have case (iii);

2) $\mathfrak{r}^{(k)}$ is contained in the radical $\mathfrak{r}$ as a proper subspace.

In the latter case, consider the ideal $\mathfrak{r}^{(k-1)}$. If its own center $Z(\mathfrak{r}^{(k-1)})$ is bigger than $\mathfrak{r}^{(k)}$, then $Z(\mathfrak{r}^{(k-1)})$ is a commutative ideal of dimension greater than 1 and we have case (i). If $Z(\mathfrak{r}^{(k-1)}) = \mathfrak{r}^{(k)}$, then $\mathfrak{r}^{(k-1)}$ is a two-step nilpotent Lie algebra with one-dimensional center, i.e., is isomorphic to the Heisenberg algebra and we have case (iii). □

It turns out that an induction step (i.e., reducing of dimension) can naturally be done in the two first cases (i) and (ii) (see below). In the third and forth cases no inductive step is needed because a complete commutative subalgebra in $S(\mathfrak{g})$ can be constructed by the argument shift method.

Consider the first case (i). Let $\mathfrak{h} \subset \mathfrak{g}$ be a commutative ideal. First of all we give a "differential" description of the polynomials $f \in \mathrm{Ann}(\mathfrak{h})$. For each $x \in \mathfrak{g}^*$, denote by $h = \pi_{\mathfrak{h}^*}(x) \in \mathfrak{h}^*$ its image under the natural projection $\pi_{\mathfrak{h}^*} : \mathfrak{g}^* \to \mathfrak{h}^*$. Consider the representation $(\mathrm{ad}\,|_{\mathfrak{h}})^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{h}^*)$ dual to the adjoint one $\mathrm{ad}\,|_{\mathfrak{h}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ and the corresponding stationary subalgebra $\mathrm{St}\,(h) \subset \mathfrak{g}$ of $h = \pi_{\mathfrak{h}^*}(x) \in \mathfrak{h}^*$.

It is easy to verify the following

Lemma 2 *If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $f \in \mathrm{Ann}(\mathfrak{h})$ if and only if $df(x) \in \mathrm{St}\,(h)$ for any $x \in \mathfrak{g}^*$.*

Proof. The condition $f \in \mathrm{Ann}(\mathfrak{h})$ means that

$$\{f, \eta\}(x) = \langle x, [df(x), \eta]\rangle = 0 \quad \text{for any} \quad \eta \in \mathfrak{h}. \tag{4}$$

Since $\mathfrak{h}$ is an ideal, this can be rewritten as

$$0 = \langle x, [df(x), \eta]\rangle = \langle h, [df(x), \eta]\rangle = -\langle (\mathrm{ad}\,|_{\mathfrak{h}})^*_{df(x)} h, \eta\rangle,$$

that is, $(\mathrm{ad}\,|_{\mathfrak{h}})^*_{df(x)} h = 0$, i.e. $df(x) \in \mathrm{St}\,(h)$, as required. □

Notice that (4) can be rewritten as $\langle \mathrm{ad}^*_{\mathfrak{h}}\, x, df(x)\rangle = 0$. In particular, we have

Corollary 1 $\mathrm{St}\,(h) = (\mathrm{ad}^*_{\mathfrak{h}}\, x)^{\perp} = \{\xi \in \mathfrak{g} \mid \langle \mathrm{ad}^*_{\mathfrak{h}}\, x, \xi\rangle = 0\}$.

Since the analysis of differentials is not always an easy task, we give another version of the above statement, which can be convenient for applications.

Corollary 2 *Let $f : \mathfrak{g}^* \to \mathbb{K}$ satisfy the condition $f(x+l) = f(x)$ for any $l \in \mathrm{St}\,(h)$, $h = \pi_{\mathfrak{h}^*}(x)$, then $f \in \mathrm{Ann}(\mathfrak{h})$.*

We now describe some "basic" elements in $\mathrm{Ann}(\mathfrak{h})$. Let $\Psi : \mathfrak{h}^* \to \mathfrak{g}$ be a polynomial map such that $\Psi(h) \in \mathrm{St}\,(h)$ for any $h \in \mathfrak{h}^*$ (among such maps there are, in particular, constant maps into $\mathfrak{h}$). In other words, Ψ is a polynomial section of the *stationary subalgebra fiber bundle* over $\mathfrak{h}^*$.

The family of such sections is endowed with the natural structure of a Lie algebra by:

$$[\Psi_1, \Psi_2](h) = [\Psi_1(h), \Psi_2(h)].$$

Consider the following polynomial function on $\mathfrak{g}^*$

$$f_\Psi(x) = \langle x, \Psi(\pi_{\mathfrak{h}^*}(x))\rangle. \tag{5}$$

Lemma 3 *The function $f_\Psi(x)$ belongs to* $\mathrm{Ann}(\mathfrak{h})$. *Moreover, the mapping $\Psi \to f_\Psi$ is a homomorphism of Lie algebras.*

Proof. We have

$$df_\Psi(x) = d\langle x, \Psi(\pi_{\mathfrak{h}^*}(x))\rangle = \Psi(\pi_{\mathfrak{h}^*}(x)) + \langle x, d\Psi(\pi_{\mathfrak{h}^*}(x))\rangle. \tag{6}$$

The first term $\Psi(\pi_{\mathfrak{h}^*}(x)) = \Psi(h)$ belongs to $\mathrm{St}\,(h)$ by definition. The second term belongs to $\mathfrak{h}$, since the section Ψ depends only on the projection $h = \pi_{\mathfrak{h}^*}(x) \in \mathfrak{h}^*$. Since $\mathfrak{h}$ is commutative, we have $\mathfrak{h} \subset \mathrm{St}\,(h)$ and, consequently, $df_\Psi(x) \in \mathrm{St}\,(h)$. Thus, $f_\Psi \in \mathrm{Ann}(\mathfrak{h})$ by Lemma 2.

Furthermore, consider two sections Ψ_1 and Ψ_2. Denoting $\langle x, d\Psi_i(\pi_{\mathfrak{h}^*}(x))\rangle$ by η_i, we have

$$\begin{aligned}
&\{f_{\Psi_1}, f_{\Psi_2}\}(x) = \langle x, [\Psi_1(h) + \eta_1, \Psi_2(h) + \eta_2]\rangle = \\
&\langle x, [\Psi_1, \Psi_2](h)\rangle + \langle x, [\Psi_1(h), \eta_2]\rangle + \langle x, [\eta_1, \Psi_2(h)]\rangle = \\
&f_{[\Psi_1,\Psi_2]}(x) + \langle h, [\Psi_1(h), \eta_2]\rangle + \langle h, [\eta_1, \Psi_2(h)]\rangle
\end{aligned}$$

The last two terms vanish since $\Psi_i(h) \in \mathrm{St}\,(h)$ and we obtain finally

$$\{f_{\Psi_1}, f_{\Psi_2}\}(x) = f_{[\Psi_1,\Psi_2]}(x).$$

In other words, the mapping $\Psi \to f_\Psi$ is a homomorphism of the algebra of sections into $\mathrm{Ann}(\mathfrak{h}) \subset S(\mathfrak{g})$, as needed. □

Lemma 4 $\mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}) = \dim \mathrm{St}\,(h) = \mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}}\, x$ *for generic* $x \in \mathfrak{g}^*$.

Proof. The inequality

$$\mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}) \le \mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}}\, x$$

is general and simply means that "the number of independent invariants cannot be greater than the codimension of a generic orbit". On the other hand, Lemma 3 explains how one can construct at least $\dim \mathrm{St}\,(h)$ algebraically independent polynomials from $\mathrm{Ann}(\mathfrak{h})$, hence

$$\mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}) \ge \dim \mathrm{St}\,(h).$$

Finally, the equality $\dim \mathrm{St}\,(h) = \mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}}\, x$ follows directly from Corollary 1. □

This statement says that $\mathrm{Ann}(\mathfrak{h})$ has sufficiently many independent polynomials and we may apply Proposition 2 (see Remark 2). In other words, a complete commutative subalgebra in $S(\mathfrak{g})$ can be obtained from any two complete commutative subalgebras $\mathcal{A}_1 \subset S(\mathfrak{h})$ and $\mathcal{A}_2 \subset \mathrm{Ann}(\mathfrak{h})$. Also notice that in our case $S(\mathfrak{h}) \subset \mathrm{Ann}(\mathfrak{h})$ so that we only need to construct a commutative subalgebra $\mathcal{A}$ which is complete in $\mathrm{Ann}(\mathfrak{h})$. In other words, we have

Proposition 3 *Let $\mathcal{A}$ be a complete commutative subalgebra in* $\operatorname{Ann}(\mathfrak{h})$*, then $\mathcal{A}$ is complete in $S(\mathfrak{g})$.*

Another important remark is that $S(\mathfrak{h})$ is contained in the center of $\operatorname{Ann}(\mathfrak{h})$ so that we may consider polynomials from $S(\mathfrak{h})$ as "new coefficients". Now we are going to explain how this idea allows us to reduce the problem to a Lie algebra of lower dimension (but over an extended field!).

Let $p = p(\eta_1, \dots, \eta_l) \in S(\mathfrak{h})$ be an arbitrary polynomial on $\mathfrak{h}^*$, where $\eta_1, \dots, \eta_l$ is a certain basis in $\mathfrak{h}$. If $\Psi : \mathfrak{h}^* \to \mathfrak{g}$ is a polynomial section of the stationary subalgebra fiber bundle, then so is $p\Psi$. Besides $[p_1\Psi_1, p_2\Psi_2] = p_1p_2[\Psi_1.\Psi_2]$. This means that elements from $S(\mathfrak{h})$ can be treated as "new coefficients" for the algebra of sections. The same is true for $\operatorname{Ann}(\mathfrak{h})$: it is a module over the ring $\mathbb{K}[\mathfrak{h}^*] = S(\mathfrak{h})$ (not only as a commutative algebra of polynomials but also as a Lie algebra). Moreover, the homomorphism of Lie algebras $\Psi \to f_\Psi$ is $\mathbb{K}[\mathfrak{h}^*]$-linear.

This observation allows us to pass to a new field of coefficients, namely $\mathbb{K}(\mathfrak{h}^*) = \operatorname{Frac} S(\mathfrak{h})$. To do this correctly we need to extend all our objects by admitting division by polynomials from $\mathbb{K}[\mathfrak{h}^*] = S(\mathfrak{h})$. Instead of $\operatorname{Ann}(\mathfrak{h})$ we consider

$$\operatorname{Ann}_{\mathrm{frac}}(\mathfrak{h}) = \left\{ \frac{f}{g} \;\middle|\; f \in \operatorname{Ann}(\mathfrak{h}), g \in S(\mathfrak{h}) \right\}$$

Analogously, instead of polynomial sections $\Psi : \mathfrak{h}^* \to \mathfrak{g}$, we consider rational ones. As above (see example in Section 3), we denote the algebra of rational sections by $L(\mathfrak{g}, (\operatorname{ad}|_{\mathfrak{h}})^*, \mathfrak{h}^*)$, and its image in $\operatorname{Ann}_{\mathrm{frac}}(\mathfrak{h})$ under the mapping $\Psi \to f_\Psi$ by $L_{\mathfrak{h}}$.

The crucial point of the proof is that all these objects $\operatorname{Ann}_{\mathrm{frac}}(\mathfrak{h})$, $L(\mathfrak{g}, (\operatorname{ad}|_{\mathfrak{h}})^*, \mathfrak{h}^*)$ and $L_{\mathfrak{h}}$ can now be treated as Lie algebras over $\mathbb{K}(\mathfrak{h}^*) = \operatorname{Frac} S(\mathfrak{h})$. The same is true for the homomorphism $\Psi \to f_\Psi$. Moreover, though the Lie algebra $L_{\mathfrak{h}}$ is infinite-dimensional over the initial field $\mathbb{K}$, it becomes finite-dimensional over $\mathbb{K}(\mathfrak{h}^*)$!

Lemma 5 $\dim_{\mathbb{K}(\mathfrak{h}^*)} L_{\mathfrak{h}} = \dim_{\mathbb{K}} \operatorname{St}(h) - \dim_{\mathbb{K}} \mathfrak{h} + 1$*, where* $\operatorname{St}(h)$ *is a generic stationary subalgebra of the representation* $(\operatorname{ad}|_{\mathfrak{h}})^* : \mathfrak{g} \to \operatorname{End}(\mathfrak{h}^*)$.

Proof. To find the dimension of $L_{\mathfrak{h}}$, we describe the kernel of the homomorphism $\Psi \to f_\Psi$. It is not hard to see that $f_\Psi = 0$ if and only if $\Psi(h) \in \operatorname{Ker}(h)$, where $\operatorname{Ker}(h) \subset \mathfrak{h}$ is the kernel of the linear functional $h \in \mathfrak{h}^*$. The dimension of the subspace of such sections Ψ over $\operatorname{Frac} S(\mathfrak{h})$ is equal to $\dim \mathfrak{h} - 1$. Taking into account that the dimension of the algebra of sections $L(\mathfrak{g}, (\operatorname{ad}|_{\mathfrak{h}})^*, \mathfrak{h}^*)$ over $\mathbb{K}(\mathfrak{h}^*)$ is equal to the dimension of a generic fiber, i. e., $\dim_{\mathbb{K}} \operatorname{St}(h)$, we immediately obtain the result. □

Thus, we have constructed a finite dimensional subalgebra $L_{\mathfrak{h}} \subset \operatorname{Ann}_{\mathrm{frac}}(\mathfrak{h})$ over the extended field $\mathbb{K}(\mathfrak{h}^*)$. Notice that its dimension is strictly less than $\dim \mathfrak{g}$ (it concides with $\dim \mathfrak{g}$ in the only case, when $\dim \mathfrak{h} = 1$ and simultaneously $\dim \operatorname{St}(h) = \dim \mathfrak{g}$, i.e. $\mathfrak{h} \subset Z(\mathfrak{g})$, but exactly this situation has been excluded from case (i), see Lemma 1).

Assume that we are able to solve our initial problem (i.e., to construct a complete commutative subalgebra) for the finite dimensional Lie algebra

$L_{\mathfrak{h}}$ in the sense of the new field $\mathbb{K}(\mathfrak{h}^*)$. It turns out that this leads us immediately to the solution of the problem for $\mathfrak{g}$ over the initial field $\mathbb{K}$. To see this, we just need to give some comments.

Let $\mathcal{A}$ be a complete commutative subalgebra in $S(L_{\mathfrak{h}})$ in the sense of $\mathbb{K}(\mathfrak{h}^*)$. Without loss of generality, we shall assume that together with any two polynomials f and g the algebra $\mathcal{A}$ contains their product fg and also contains all the constants, i.e. elements from $\mathbb{K}(\mathfrak{h}^*)$.

Notice first of all that $S(L_{\mathfrak{h}})$ can naturally be considered as a subalgebra in $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h})$, since $L_{\mathfrak{h}} \subset \mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h})$. Therefore any commutative subalgebra $\mathcal{A} \subset S(L_{\mathfrak{h}})$ can be treated as a commutative subalgebra in $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h})$.

Thus, we can look at $\mathcal{A}$ from two different points of view: either as a subalgebra in $S(L_{\mathfrak{h}})$ in the sense of the extended field $\mathbb{K}(\mathfrak{h}^*)$, or a subalgebra in $S(L_{\mathfrak{h}})$ in the sence of the initial field $\mathbb{K}$ (and then both $\mathcal{A}$ and $S(L_{\mathfrak{h}})$ are considered as subalgebras in $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h})$).

We have assumed that $\mathcal{A}$ is complete in $S(L_{\mathfrak{h}})$ in the sense of $\mathbb{K}(\mathfrak{h}^*)$. Will it be complete in $S(L_{\mathfrak{h}})$ in the sense of the initial field $\mathbb{K}$? It is not hard to see that the answer is positive.

The next question: is this algebra $\mathcal{A}$ complete in $\mathrm{Ann}_{\mathrm{frac}}\,\mathfrak{h})$? The answer is obviously positive because at a generic point $x \in \mathfrak{g}^*$, the subspaces in $\mathfrak{g}$ generated by the differentials of functions from $S(L_{\mathfrak{h}})$ and from $\mathcal{A}$ are exactly the same (both of them coincide with $\mathrm{St}\,(h)$, see Lemma 2).

The last difficulty is that the functions from $\mathcal{A}$ are not polynomial, but rational. More precisely, they are all of the form $\dfrac{f}{g}$, where $g \in \mathbb{K}(\mathfrak{h}^*)$. But together with $\dfrac{f}{g}$ this subalgebra contains both f and g separately. Therefore, the difficulty can be avoided just by taking the "polynomial" part of $\mathcal{A}$, or simply by multiplying each fraction by its denominator. After this operation we obtain a certain subalgebra $\mathcal{A}_{\mathrm{pol}}$ in $\mathrm{Ann}(\mathfrak{h})$ which is obviously commutative and complete (just because the number of independent functions remains the same). In other words, after "polynomialization" $\mathcal{A} \to \mathcal{A}_{\mathrm{pol}}$ any complete commutative subalgebra $\mathcal{A} \subset S(L_{\mathfrak{h}})$ remains complete in $\mathrm{Ann}(\mathfrak{h})$. Taking into account Proposition 3, we come to the following conclusion.

Proposition 4 *If the Mischenko-Fomenko conjecture holds for $L_{\mathfrak{h}}$ over $\mathbb{K}(\mathfrak{h}^*)$, then it holds for $\mathfrak{g}$ over the initial field $\mathbb{K}$.*

Thus, in case (i) from Lemma 1, the problem is reduced to a Lie algebra of smaller dimension.

Let us now consider the second case. Suppose that algebra $\mathfrak{g}$ has an ideal isomorphic to the Heisenberg algebra $\mathfrak{h}_m$, and the center of $\mathfrak{h}_m$ coincides with the center of $\mathfrak{g}$. Recall the structure of the Heisenberg algebra : $\mathfrak{h}_m$ splits into the direct sum of a subspace V of dimension $2m$ and the one-dimensional center $Z(\mathfrak{h}_m)$ generated by a vector e. For two arbitrary elements $\xi_1, \xi_2 \in V$, their commutator is defined by

$$[\xi_1, \xi_2] = \omega(\xi_1, \xi_2)e,$$

where ω is a symplectic form on V.

First we notice several useful properties of $\mathfrak{g}$.

Lemma 6 *Let $\mathfrak{h}_m \subset \mathfrak{g}$ be an ideal. Then there exists a subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{b} \oplus V$ and $\mathfrak{b} \cap \mathfrak{h}_m = Z(\mathfrak{h}_m)$. Besides, the subspace $V \subset \mathfrak{h}_m$ is invariant under the adjoint action of $\mathfrak{b}$ and $\mathfrak{b}$ acts on $\mathfrak{h}_m$ by symplectic transformations.*

Proof. We define $\mathfrak{b}$ in the following way:

$$\mathfrak{b} = \{\xi \in \mathfrak{g} \mid \mathrm{ad}_\xi(V) \subset V.$$

Obviously, $\mathfrak{b}$ is a subalgebra in $\mathfrak{g}$. Let us check that any element $\xi \in \mathfrak{g}$ can be uniquely presented in the form $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \mathfrak{b}$, $\xi_2 \in V$.

For $v \in V$, we take $[\xi, v] \in \mathfrak{h}_m$ and decompose it with respect to the subspaces V and $Z(\mathfrak{h}_m)$:

$$[\xi, v] = \eta_1 + \eta_2, \quad \eta_1 \in V, \eta_2 \in Z(\mathfrak{h}_m).$$

Since the center $Z(\mathfrak{h}_m)$ is one-dimensional η_2 can be presented as $\eta_2 = l_\xi(v)e$, where $l_\xi : V \to \mathbb{K}$ is a certain linear functional. Since V is endowed with a non-degenerate symplectic structure, this functional can be taken in the form $l_\xi(v) = \omega(\xi_2, v)$, where $\xi_2 \in V$ is a certain element which is uniquely defined by ξ. It is easy to see that $\xi - \xi_2$ leaves the space V invariant:

$$[\xi - \xi_2, v] = \eta_1 + \eta_2 - [\xi_2, v] = \eta_1 + \omega(\xi_2, v)e - \omega(\xi_2, v)e = \eta_1 \in V.$$

Thus, $\mathfrak{g} = \mathfrak{b} \oplus V$ is a direct sum of the subspaces. Also it is easy to see that, $\mathfrak{b} \cap \mathfrak{h}_m = Z(\mathfrak{h}_m)$.

We need finally to prove that the representation $\mathrm{ad} : \mathfrak{b} \to \mathrm{End}(V)$ is symplectic, i.e., each transformation $\mathrm{ad}_\beta : V \to V$ is an element of the symplectic Lie algebra $sp(V, \omega)$ for any $\beta \in \mathfrak{b}$.

To this end, we use the Jacobi identity. We have:

$$\mathrm{ad}_\beta[v_1, v_2] = [\mathrm{ad}_\beta v_1, v_2] + [v_1, \mathrm{ad}_\beta v_2] = \omega(\mathrm{ad}_\beta v_1, v_2) + \omega(v_1, \mathrm{ad}_\beta v_2).$$

On the other hand, $[v_1, v_2]$ belongs to the center, therefore $\mathrm{ad}_\beta[v_1, v_2] = 0$. Thus,

$$\omega(\mathrm{ad}_\beta v_1, v_2) + \omega(v_1, \mathrm{ad}_\beta v_2) = 0,$$

which is equivalent to the symplecticity of the represenation $\mathrm{ad} : \mathfrak{b} \to gl(V)$. □

Remark 3 It is not hard to verify that $\mathrm{ind}\, \mathfrak{b} = \mathrm{ind}\, \mathfrak{g}$. The proof is straightforward. The same result will, however, follow from our consideration below.

Following our general idea we need to consider $\mathfrak{h}_m$ and its annihilator $\mathrm{Ann}(\mathfrak{g}_m)$. It turn out that the functions from $\mathrm{Ann}(\mathfrak{g}_m)$ admit a very natural description.

For any element $\beta \in \mathfrak{b}$ we define a quadratic polynomial

$$f_\beta(x) = \langle \beta, x \rangle \langle e, x \rangle + \frac{1}{2} \langle \omega^{-1}\big((\mathrm{ad}_\beta)^* \pi(x)\big), x \rangle. \tag{7}$$

Here $\pi : \mathfrak{g}^* \to V^*$ is the natural projection, $(\mathrm{ad}_\beta)^* : V^* \to V^*$ is the operator dual to $\mathrm{ad}_\beta : V \to V$, ω — is a symplectic structure on V treated as a mapping from V to V^* so that ω^{-1} is an inverse operator from V^* to V, e is a basis element of the center.

Lemma 7 $f_\beta \in \mathrm{Ann}(\mathfrak{g}_m)$.

Proof. We need to verify the following identity

$$\langle x, [df_\beta(x), \eta]\rangle = 0$$

for any $\eta \in \mathfrak{h}_m$, $x \in \mathfrak{g}^*$.

Compute the differential of f_β. First notice that the quadratic form $\langle Cx, y\rangle = \langle \omega^{-1}\big((\mathrm{ad}_b)^*\pi(x)\big), y\rangle$ is symmetric, therefore $d\langle Cx, x\rangle = 2Cx$. Hence

$$df_\beta(x) = \beta\langle e, x\rangle + e\langle \beta, x\rangle + \omega^{-1}\big((\mathrm{ad}_\beta)^*\pi(x)\big).$$

Then for arbitrary $\eta \in \mathfrak{h}_m$ we have:

$$\begin{aligned}
&\langle [df_\beta(x), \eta], x\rangle = \\
&\langle [\beta\langle e, x\rangle + e\langle \beta, x\rangle + \omega^{-1}\big((\mathrm{ad}_\beta)^*\pi(x)\big), \eta], x\rangle \\
&\langle e, x\rangle\langle \mathrm{ad}_\beta\, \eta, x\rangle + \omega\big(\omega^{-1}\big((\mathrm{ad}_\beta)^*\pi(x), \eta\big)\langle e, x\rangle = \\
&\langle e, x\rangle\langle \mathrm{ad}_\beta\, \eta, x\rangle + \langle (\mathrm{ad}_\beta)^*\pi(x), \eta\rangle)\langle e, x\rangle = \\
&\langle e, x\rangle\langle \mathrm{ad}_\beta\, \eta, x\rangle - \langle \pi(x), \mathrm{ad}_\beta\, \eta\rangle)\langle e, x\rangle = \\
&\langle e, x\rangle\langle \mathrm{ad}_\beta\, \eta, x\rangle - \langle x, \mathrm{ad}_\beta\, \eta\rangle)\langle e, x\rangle = 0. \ \square
\end{aligned}$$

The next statement is an analog of Lemma 4.

Lemma 8 $\mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}_m) = \dim \mathfrak{b} = \mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}_m} x$ *for generic* $x \in \mathfrak{g}$*.*

Proof. Here by ad^* we denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. However for the subalgebra $\mathfrak{h}_m$ we may consider the coadjoint action on its own dual space $\mathfrak{h}_m^*$. Denote this action by $\tilde{ad}^*$ for a moment. Consider two subspaces $\mathrm{ad}^*_{\mathfrak{h}_m} x$ and $\tilde{\mathrm{ad}}^*_{\mathfrak{h}_m} h$, where x is generic in $\mathfrak{g}^*$ and h is generic in $\mathfrak{h}_m$. It is a general and obvious fact that

$$\dim \mathrm{ad}^*_{\mathfrak{h}_m} x \geq \dim \tilde{\mathrm{ad}}^*_{\mathfrak{h}_m} h.$$

But $\dim \tilde{\mathrm{ad}}^*_{\mathfrak{h}_m} h = \dim \mathfrak{h}_m - \mathrm{ind}\, h_m = 2m + 1 - 1 = 2m$ so that

$$\mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}_m} x \leq \dim \mathfrak{g} - 2m = \dim \mathfrak{b}$$

On the other hand, Lemma 7 gives us an explicit formula for $\dim \mathfrak{b}$ independent polynomials from $\mathrm{Ann}(\mathfrak{h}_m)$ and, consequently,

$$\dim \mathfrak{b} \leq \mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}_m)$$

Taking into account the general inequality $\mathrm{tr.deg}\,\mathrm{Ann}(\mathfrak{h}_m) \leq \mathrm{codim}\,\mathrm{ad}^*_{\mathfrak{h}_m} x$ we come to the desired conclusion. $\square$

This lemma asserts, in particular, that $\mathrm{Ann}(\mathfrak{h}_m)$ has sufficiently many independent functions so that we may apply Proposition 2 (see Remark 2). In other words, we have

Proposition 5 *Let $\mathcal{A}$ be a complete commutative subalgebra in* $\mathrm{Ann}(\mathfrak{h}_m)$ *and $\mathcal{B}$ be a complete commutative subalgebra in $S(\mathfrak{h}_m)$, then $\mathcal{A} + \mathcal{B}$ is complete in $S(\mathfrak{g})$.*

As we see from Lemma 7, the subalgebra b and the annihilator $\mathrm{Ann}(\mathfrak{h}_m)$ are closely related. The following construction explains this relationship more explicitly. Instead of f_β it will be more convenient to consider the rational function of the form: $\tilde{f}_\beta(x) = f_\beta(x)/\langle e, x\rangle$.

Notice the following remarkable fact which can be verified by a straightforward computation.

Lemma 9 *The map $\beta \to \tilde{f}_\beta$ is an embedding (monomorphism) of $\mathfrak{b}$ into Frac$(S(\mathfrak{g}))$.*

The further construction follows the same idea as in the first case (i). First we need to admit division by the central elements $g \in S(Z(\mathfrak{g}))$. Notice that these elements are just polynomials of one variable e, generator of the center $Z(\mathfrak{g})$. Thus , we consider

$$\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m) = \left\{ \frac{f}{g} \;\middle|\; f \in \mathrm{Ann}(\mathfrak{h}), g \in S(Z(\mathfrak{g})\right\}$$

The map $\beta \to \tilde{f}_\beta$ generates an embedding of b and, consequently, of $S(\mathfrak{b})$ into $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m)$.

For applications, it is convenient to rewrite the embedding in dual terms. Let $f : \mathfrak{b}^* \to \mathbb{K}$ be a polynomial function on $\mathfrak{b}^*$. Introduce a new function $\tilde{f} : \mathfrak{g}^* \to \mathbb{K}$ by letting

$$\tilde{f}(x) = \tilde{f}(b + v) = f(b + 1/2\langle e, b\rangle \cdot l_v)$$

where l_v denotes a linear functional on $\mathfrak{b}$ defined by

$$l_v(\beta) = \langle \omega^{-1}((\mathrm{ad}_\beta)^* v), v\rangle$$

and $x = b + v$ is the decomposition dual to $\mathfrak{g} = \mathfrak{b} + V$.

The following statement is just a reformulation of Lemmas 7 and 9.

Lemma 10 *The map $f \to \tilde{f}$ is an embedding of $S(\mathfrak{b})$ into $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m)$.*

Now it is easy to see that the construction of a complete commutative subalgebra in $S(\mathfrak{g})$ is naturally reduced to the same problem for $S(\mathfrak{b})$.

Indeed, suppose we have a complete commutative subalgebra in $S(\mathfrak{b})$. As before, we assume that this algebra is closed with respect to usual multiplication and contains $S(Z(\mathfrak{g}))$.

Consider its image $\tilde{\mathcal{A}}$ in $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m)$ under the mapping $f \to \tilde{f}$. We claim that $\mathcal{A}$ is complete in $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m)$. This follows immediately from the fact that at a generic point, the subspaces in $\mathfrak{g}$ generated by the functions from $\mathrm{Ann}_{\mathrm{frac}}(\mathfrak{h}_m)$ and by the functions of the form $\tilde{f}$, where $f \in S(\mathfrak{b})$ exactly coincide (since they have the same dimension $\dim \mathfrak{b}$, see Lemma 8). Finally, to obtain *polynomial* complete commutative subalgebra in $\mathrm{Ann}(\mathfrak{h}_m)$, we just take the polynomial part $\tilde{\mathcal{A}}_{\mathrm{pol}}$ of $\tilde{\mathcal{A}}$, see above for details.

Proposition 6 *If $\mathfrak{b}$ satisfies the Mischenko-Fomenko conjecture, then so does $\mathfrak{g}$.*

Thus, we have shown that in cases (i) and (ii), the proof of the Mischenko-Fomenko conjecture can be reduced to the algebra $\mathfrak{b}$ of smaller dimension. The induction argument completes the proof of Theorem 1.

Notice that the proof is constructive: if we have a complete commutative subalgebra in $S(L_\mathfrak{h})$ or in $S(\mathfrak{b})$, we get a complete commutative subalgebra in $S(\mathfrak{g})$ by using rather simple explicit formulae.

5 Examples

In this section we show how the above construction works by studying several examples. We consider the semidirect sums:

1) $so(n) +_\phi \mathbb{R}^n$,

2) $sp(2n) +_\phi \mathbb{R}^{2n}$,

3) $gl(n) +_\phi \mathbb{R}^n$, with respect to standard representations.

Recall that our construction is a step-by-step procedure. At each step we reduce the dimension of the Lie algebra under consideration until we come to either one-dimensional or semisimple Lie algebra. The first case is the simplest. After one step we come to a semisimple Lie algebra and then apply the argument shift method. The second Lie algebra $sp(2n) +_\phi \mathbb{R}^{2n}$ needs two steps (of two different types corresponding to cases (1) and (2) from Lemma 1). The affine Lie algebra $gl(n)+_\phi\mathbb{R}^n$ is "more complicated": we never come to the semisimple algebra, but have to make n steps before we finish with the trivial Lie algebra.

We first discuss several general facts. Consider a semidirect sum $\mathfrak{g} = \mathfrak{k} +_\rho V$ of a Lie algebra $\mathfrak{k}$ and a commutative ideal V. Its dual space is naturally identified with $\mathfrak{k}^* + V^*$ and we shall represent elements of $\mathfrak{g}^*$ as pairs (M, v), where $M \in \mathfrak{k}^*$, $v \in V^*$.

According to our general approach, we are going to make "reduction" with respect to V as a commutative ideal $\mathfrak{h}$ from Lemma 1, case 1. By $\mathrm{St}_{\rho^*}(v)$ we denote the stationary subalgebra of $v \in V^*$ with respect to the dual representation $\rho^* : \mathfrak{k} \to \mathrm{End}(V^*)$. It is easy to see that the stationary subalgebra $\mathrm{St}(v)$ considered in Lemma 2 is just the semidirect sum of $\mathrm{St}_{\rho^*}(v)$ and the ideal V. The following statement is a reformulation of Corollary 2 in this particular case.

Lemma 11 *Let $f : \mathfrak{g}^* \to \mathbb{R}$ satisfy the following condition:*

$$f(M, v) = f(M + L, v) \quad \textit{for any} \quad L \in \mathrm{St}_{\rho^*}(v)^\perp \subset \mathfrak{k}^*. \tag{8}$$

Then $f \in \mathrm{Ann}(V)$.

Condition (8) has a very natural geometrical meaning. Namely, if we think of v as a parameter, then $f(M, v)$ can naturally be considered as a function on $\mathrm{St}_{\rho^*}(v)^*$. In particular, this function can be presented in the form $f(M, v) = f_v(\pi(M))$, where $\pi : \mathfrak{k}^* \to \mathrm{St}_{\rho^*}(v)^*$ denotes the natural projection.

Lemma 12 *Let $f(M, u)$ and $g(M, u)$ satisfy* (8)*. Then*

$$\{f(M, v), g(M, v)\} = \{f_v(\pi(M)), g_v(\pi(M))\}_{\mathrm{St}_{\rho^*}(v)},$$

where the latter is the Poisson-Lie bracket on $\mathrm{St}_{\rho^*}(v)^*$.

The proof of this statement is, if fact, similar to that of Lemma 3 and is based on the simple fact that $df(M,u) = (X,\eta) \in \mathfrak{g}$ where $\eta \in V$, $X \in \mathrm{St}_{\rho^*}(v) \subset \mathfrak{k}$.

According to the general concept, the construction of a complete commutative subalgebra in $S(\mathfrak{g})$ is reduced to the same problem for $\mathrm{Ann}(V)$. The next statement describes this reduction explicitly.

Lemma 13 *Consider a set of polynomials $f_1(M,v),\dots,f_l(M,v)$ satisfying* (8)*. Suppose that for generic $v \in V$ they commute as functions on $\mathrm{St}_{\rho^*}(v)^*$ and form a complete commutative set in $S(\mathrm{St}_{\rho^*}(v))$. Then*

$$\{f_1,\dots,f_l\} \cup V$$

is a complete commutative set in $S(\mathfrak{g})$.

We now pass to the examples. Consider the Lie algebra $\mathfrak{g} = e(n) = so(n) +_\phi \mathbb{R}^n$ (i.e., the Lie algebra of the isometry group of the Euclidean space). The dual space $e(n)^*$ is identified with $e(n)$ by means of the scalar (non-invariant!) product $\langle (M_1,v_1),(M_2,v_2)\rangle = \mathrm{Tr}\, M_1M_2 + \langle v_1,v_2\rangle$.

For generic $v \in \mathbb{R}^n$, the stationary subalgebra of the standard representation of $so(n)$ is isomorphic to $so(n-1)$. This stationary subalgebra depends on v as a parameter and is semisimple. Thus, a complete commutative set can be constructed by the argument shift method. According to Lemma 13 we need to construct a set of functions $f_1(M,v),\dots,f_k(M,v)$ such that for each (generic) v these functions becomes "the shifts of invariants" on the stationary subalgebra of v. As such functions we may consider, for instance,

$$f_{\lambda,k}(M,v) = \mathrm{Tr}\big(\mathrm{pr}_v(M+\lambda B)\big)^k$$

where $\mathrm{pr}_v : so(n) = so(n)^* \to \mathrm{St}_{\phi^*}(v) = \mathrm{St}_{\phi^*}(v)^*$ is the orthogonal projection. It is not hard to see that this projection is given by

$$\mathrm{pr}_v(M) = M - \frac{1}{|v|^2}\left(v \otimes (Mv)^\top - Mv \otimes v^\top\right).$$

The above functions are not polynomial, but rational. This problem, however, can easily be avoided by replacing $\mathrm{pr}_{\mathrm{St}\,(v)}$ with the map

$$|v|^2 \cdot \mathrm{pr}_v : so(n) \to \mathrm{St}\,(v)$$

$$|v|^2 \cdot \mathrm{pr}_v(M) = |v|^2 M - v \otimes (Mv)^\perp + Mv \otimes v^\perp,$$

which is quadratic in v (and linear in M).

As a result we obtain a family of commuting polynomials

$$\tilde f_{k,\lambda}(M,v) = \mathrm{Tr}\left(|v|^2\, \mathrm{pr}_v(M+\lambda B)\right)^k.$$

The following statement is a particular case of Lemma 13. Let $v_i = \langle v, e_i\rangle$ be coordinate linear functions on $\mathbb{R}^n$ with respect to a certain basis $e_1,\dots,e_n$.

Theorem 4 [27] *The functions*

$$v_1,\dots,v_n \quad \textit{and} \quad \tilde f_{k,\lambda}(M,v), \quad k = 2,4,\dots,[n-1], \ \lambda \in \mathbb{R},$$

generate a complete commutative subalgebra in $S\big(e(n)\big)$.

Remark 4 The above construction was studied by A.S. Ten in his diploma work [27] two years before Sadetov's proof. In fact, Ten proved this result for any semidirect sum $\mathfrak{k} +_\rho V$ if $\mathfrak{k}$ is compact. The compactness, however, can be easily replaced by the assumption that the generic stationary subalgebra of the dual representation $\rho^* : \mathfrak{k} \to \mathrm{End}(V^*)$ is semisimple.

The next example is the semidirect product $sp(2n)+_\phi\mathbb{R}^{2n}$ with respect to the standard representation. As above, the elements of $sp(2n) +_\phi \mathbb{R}^{2n}$ are presented as pairs (M, u), where $M \in sp(2n)$, $u \in \mathbb{R}^{2n}$. The dual space is identified with the algebra by

$$\langle (M_1, v_1), (M_2, v_2)\rangle = \mathrm{Tr}\, M_1 M_2 + \Omega(v_1, v_2),$$

where Ω is a symplectic form on $\mathbb{R}^{2n}$.

It is easy to see that the generic stationary subalgebra $\mathrm{St}_{\,\phi^*}(v)$ is not semisimple as in the previous case, but isomorphic to the semidirect sum $sp(2n-2) + \mathfrak{h}_{n-1}$, where $\mathfrak{h}_{n-1}$ is a Heisenberg ideal. In turn, $\mathfrak{h}_{n-1}$ is decomposed into $(2n-2)$-dimensional symplectic space V and one-dimensional center Z. Such a decomposition is not uniquely defined. To make the choice unique, we choose another element $a \in \mathbb{R}^{2n}$ such that $\Omega(a, v) \neq 0$. After this the subalgebra $sp(2n-2) \subset \mathrm{St}_{\,\phi^*}(v)$ is defined to be the common stationary subalgebra for v and a

$$\mathrm{St}_{\,\phi^*}(v, a) = \{A \in sp(2n) \mid \phi^*(A)a = \phi^*(A)v = 0\}, \qquad (9)$$

the space V is formed by matrices

$$C_\xi = v \otimes (\Omega\xi)^\top + \xi \otimes (\Omega v)^\top \qquad (10)$$

where ξ belongs to the $(2n-2)$-dimensional subspace

$$\langle v, a\rangle^\Omega = \{\xi \in \mathbb{R}^{2n} \mid \Omega(\xi, a) = \Omega(\xi, v) = 0\},$$

and the center Z is generated by the matrix

$$C_0 = v \otimes (\Omega v)^\top \qquad (11)$$

Here $\otimes$ denotes usual matrix multiplication, if we think of v as a column and of $(\Omega v)^\top$ as a row, at the same time $\otimes$ is the tensor product of a vector and a covector.

We now apply the general approach to $\mathrm{St}_{\,\phi^*}(v) = sp(2n-2) + \mathfrak{h}_{n-1}$ thinking of v as a parameter. A complete commutative family for $\mathrm{St}_{\,\phi^*}(v)$ consists of two parts. One is a complete commutative family for the Heisenberg ideal $\mathfrak{h}_{n-1}$. The other is formed by the shifts of Ad-invariants of $sp(2n-2)$ transmitted into $S(\mathrm{St}_{\,\phi^*}(v))$ by means of Lemma 10.

The functions corresponding to the Heisenberg ideal are (see (10), (11)):

$$e(M, v) = \mathrm{Tr}\, MC_0 = \mathrm{Tr}\, M\, v \otimes (\Omega v)^\top = \Omega(v, Mv)$$

and

$$\mathrm{Tr}\, MC_\xi = \mathrm{Tr}\, M(v \otimes (\Omega\xi)^\top + \xi \otimes (\Omega v)^\top) = 2\Omega(Mv, \xi)$$

If we want them to commute, then ξ must belong to a certain $(n-1)$-dimensional Lagrangian subspace in $\langle v, a\rangle^\Omega$. For instance, we may take

ξ of the form $\xi = \zeta\Omega(u,a) - a\Omega(\zeta,v)$, where ζ belongs to a certain fixed Lagrangian subspace in $\mathbb{R}^{2n}$ that contains a. In other words, as commuting functions we can take

$$f_\zeta(M,v) = \Omega(Mv, \zeta\Omega(v,a) - a\Omega(\zeta,v)) = \begin{vmatrix} \Omega(v,\zeta) & \Omega(v,a) \\ \Omega(Mv,\zeta) & \Omega(Mv,a) \end{vmatrix}$$

Finally, the shifts of Ad-invariants of $sp(2n-2) = \mathrm{St}_{\phi^*}(v,a)$ take the following form (after being transmitted into $S(\mathrm{St}_{\phi^*}(v))$ by Lemma 9 and lifted into $S(sp(2n) +_\phi \mathbb{R}_{2n})$:

$$f_{k,\lambda}(M,v) = \mathrm{Tr}\left(\mathrm{pr}_{v,a}\big(\Omega(Mv,v)M + Mv\otimes(\Omega Mv)^\top + \lambda B\big)\right)^k$$

It can be checked that the projection $\mathrm{pr}_{v,a}$ is given by

$$\begin{aligned}\mathrm{pr}_{v,a}(M) = M - \quad & \Omega(v,a)^{-1}(Ma\otimes(\Omega v)^\top - v\otimes(\Omega Ma)^\top) + \\ & \Omega(v,a)^{-2}\Omega(Ma,a)v\otimes(\Omega v)^\top - \\ & \Omega(v,a)^{-1}(Mv\otimes(\Omega a)^\top - a\otimes(\Omega Mv)^\top) + \\ & \Omega(v,a)^{-2}\Omega(Mv,v)a\otimes(\Omega a)^\top + \\ & \Omega(v,a)^{-2}\Omega(Mv,a)(a\otimes(\Omega v)^\top + v\otimes(\Omega a)^\top)\end{aligned}$$

Thus, to avoid rational functions we replace $f_{k,\lambda}(M,v)$ by

$$\tilde f_{k,\lambda}(M,v) = \mathrm{Tr}\left(\Omega(v,a)^2\,\mathrm{pr}_{v,a}\big(\Omega(Mv,v)M + Mv\otimes(\Omega Mv)^\top + \lambda B\big)\right)^k$$

Here is the final statement.

Theorem 5 *The following functions generate a complete commutative subalgebra in $S(sp(2n) +_\rho \mathbb{R}^{2n})$:*

1) $v_1, v_2, \dots, v_{2n}$ (coordinate functions on $\mathbb{R}^{2n}$);

2)$f_\zeta(M,v)$, where ζ belongs to a certain Lagrangian subspace in $\mathbb{R}^{2n}$ that contains a;

3) $e(M,v)$, the function corresponding to the center of $\mathrm{St}_{\rho^*}(v)$;

4) $\tilde f_{k,\lambda}(M,v)$, $k = 2,4,\dots,2n$, $\lambda\in\mathbb{R}$.

The last example is the affine Lie algebra $\mathfrak{aff}_n = gl(n,\mathbb{R}) + \mathbb{R}^n$.

Once again we consider $V = \mathbb{R}^n$ as a commutative ideal and follow our general approach. The stationary subalgebra of any non-zero element $v\in V^*$ with respect to the Ad*-action of $\mathfrak{aff}_n$ on V^* is isomprphic to $\mathfrak{aff}_{n-1} + \mathbb{R}^n$, where $\mathfrak{aff}_{n-1} = \mathfrak{aff}_{n-1}(v) = gl(n-1) + \mathbb{R}^{n-1} \subset \mathfrak{k} = gl(n)$ is the stationary subalgebra of v with respect to the natural action of $gl(n)$ on V^*. Thus, on the second step of the procedure, we have to deal again with the affine algebra (of smaller dimension) which depends on u as a parameter. It turns out that repeating this procedure step by step, we come to the following set of commuting functions.

Let $\xi_1, \xi_2, \dots, \xi_n \subset V = \mathbb{R}^n$. For definiteness, we think of v as a row, and of ξ_1 as a column. The functions corresponding to the commutative ideal $\mathbb{R}^n$ are:

$$f_{\xi_1}(M,v) = (\xi_1, v)$$

The functions which correspond to the commutative ideal in the stationary subalgebra $\mathrm{St}\,(v) = gl(n-1) + \mathbb{R}^{n-1}$ take the form

$$f_{\xi_1,\xi_2}(M,v) = \begin{vmatrix} (\xi_1,v) & (\xi_2,v) \\ (\xi_1,vM) & (\xi_2,vM) \end{vmatrix}$$

Analogously, on the kth step we obtain the functions.

$$f_{\xi_1,\dots,\xi_k}(M,v) = \det(a_{ij}),$$

where $a_{ij} = (\xi_j, vM^{i-1})$.

Theorem 6 *The functions $f_{\xi_1,\dots,\xi_k}(M,v)$, $\xi_i \in \mathbb{R}^n$, $k = 1,\dots,n-1$, commute for any values of parameters, i.e.:*

$$\{f_{\xi_1,\dots,\xi_l}, f_{\tilde{\xi}_1,\dots,\tilde{\xi}_k}\} = 0,$$

and generate a complete commutative subalgebra in $S(\mathfrak{aff}_n)$.

The proof can be obtained by noticing that if we fix v, we obtain the collection of functions on $\mathrm{St}\,(v) = \mathfrak{aff}_{n-1}$ just of the same form as the initial functions, i.e. of the form $f_{\eta_1,\dots,\eta_k}$ where η_i are all orthogonal to the (co)vector v. It is worth to notice that $\mathrm{St}\,(v) = \mathfrak{aff}_{n-1}$ can be naturally interpreted as an affine algebra related to the "orthogonal" complement to v, i.e. the subspace $\{\eta \in \mathbb{R}^n \mid (\eta, v) = 0\}$, $v \in (\mathbb{R}^n)^*$. After this remark, the proof is obtained by induction.

6 Two open questions in conclusion

The Mishchenko-Fomenko conjecture has several natural generalizations. Actually, the existence of a complete commutative subalgebra is a very important property to be studied for any polynomial Poisson algebra. One of the most important examples of polynomial Poisson algebra are those of the form $\mathrm{Ann}(\mathfrak{h})$, where $\mathfrak{h}$ is a certain subalgebra of a finite dimensional Lie algebra $\mathfrak{g}$.

In the particular case of compact Riemannian homogeneous spaces G/H, the existence of a complete commutative subalgebra in $\mathrm{Ann}(\mathfrak{h})$ would guarantee the integrability of the geodesic flow on G/H by means of polynomial integrals (here $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively).

Question 1. Does $\mathrm{Ann}(\mathfrak{h})$ always admit a complete commutative subalgebra?

According to the strong definition of integrability, in addition to commutativity and completeness of first integrals $f_1,\dots,f_k \in S(\mathfrak{g})$ in the sense Definition 1, one should require the completeness of each Hamiltonian vector field $X_{f_i}(x) = \mathrm{ad}^*_{df_i(x)}\, x$ in the sense that the corresponding Hamiltonian flow $\sigma^t_{X_{f_i}}$ is well defined for all $t \in (-\infty, +\infty)$.

Question 2. Consider the complete commutative subalgebra $\mathcal{A} \subset \mathfrak{g}$ constructed in Theorem 1 (see proof in Section 4). Are the Hamiltonian flows of $f \in \mathcal{A}$ complete?

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