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Compatible Poisson Brackets on Lie Algebras

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Abstract—We discuss the relationship between the representation of an integrable system as an L-A-pair with a spectral parameter and the existence of two compatible Hamiltonian representations of this system. We consider examples of compatible Poisson brackets on Lie algebras, as well as the corresponding integrable Hamiltonian systems and Lax representations.

Key words: *compatible Poisson brackets, compatible Hamiltonian representation, Lax representation, integrable Hamiltonian system, bi-Hamiltonian vector field, Lie algebra.*

1. INTRODUCTION

1.1. Compatible Poisson brackets and Hamiltonian systems

Many completely integrable Hamiltonian systems arising in mechanics, mathematical physics, and geometry have the remarkable property of being bi-Hamiltonian, i.e., they are Hamiltonian systems with respect to two different Poisson structures at once (e.g., see [1–10]). Very often, these structures are mutually compatible, and the system in question is Hamiltonian with respect to any of their linear combinations (with constant coefficients).

On the other hand, most integrable Hamiltonian systems can be written in Lax form

$$\dot{L} = [L, A],$$

where L and A are square matrices of phase variables (they are said to form an *L-A-pair*). The possibility of introducing an arbitrary (spectral) parameter in the elements of L and A (this parameter is not contained in the equations of motion) is typical of integrable systems.

In this paper, we study relations between the possibility of representing an integrable system as an L-A-pair with a spectral parameter and the fact that this system has two compatible Hamiltonian representations. In some sense, these relations provide a geometric and mechanical justification of formal algebraic constructions in the L-A-pair method, since the Poisson structure (as well as the first integrals) is a tensor invariant of the equations of motion and is meaningful as a (tensor) law of conservation. We note that other justifications of the possibility of introducing a spectral parameter in the L-A-pair, for instance, the R-matrix method [11], have only a formal algebraic meaning.

Let us recall the necessary definitions. Let $A = (A^{ij})$ be a skew-symmetric tensor field of type $(2, 0)$ on a manifold M. This field determines the following natural operation on the space of smooth functions:

$$\{f, g\} = A^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

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Definition 1. The *Poisson structure* on a smooth manifold M is defined to be a skew-symmetric tensor field A^{ij} of type $(2,0)$ such that the operation $\{\ ,\ \}$ determined by this field satisfies the Jacobi identity

$$\{f,\{g,h\}\}+\{f,\{g,h\}\}+\{f,\{g,h\}\}=0 \qquad \text{for any } f,g,h\in C^\infty(M).$$

In this situation, the space of smooth functions has the natural structure of an infinite-dimensional Lie algebra, and the operation $\{\ ,\ \}$ is called the *Poisson bracket.*

Definition 2. Let f be a smooth function. A vector field of the form

$$(\operatorname{sgrad} f)^j = A^{ij}\frac{\partial f}{\partial x^i}$$

is said to be *Hamiltonian.* Then the function f is called a *Hamiltonian.* The invariant definition states: $\operatorname{sgrad} f$ is a vector field such that any smooth function g satisfies the identity

$$\operatorname{sgrad} f(g)=\{f,g\}.$$

Definition 3. Two Poisson structures A and B are said to be *compatible* if their sum (or, which is the same, any arbitrary linear combination of A and B with constant coefficients) is also a Poisson structure (a similar definition holds for the Poisson brackets).

Comment. The nontrivialiaty of the compatibility condition is in that the sum of two Poisson brackets also satisfies the Jacobi identity. This condition can be rewritten analytically as follows: it is equivalent to the fact that the so-called Schouten bracket $\{\{A,B\}\}$ of the Poisson structures A and B is zero (see [3, 12]). However, for the Poisson structure A, the Jacobi identity itself can be written in these terms as $\{\{A,A\}\}=0$.

Example 1. Any two constant Poisson brackets are compatible.

Naturally, each Poisson bracket can be assigned its rank. This is simply the rank of the matrix A^{ij} at a point in general position. In what follows, we shall study only real-analytic Poisson brackets. Hence here we do not always assume that the points in general position form an open everywhere dense subset in M.

If $\operatorname{rank} A=\dim M$, then the Poisson bracket is said to be *nondegenerate* (almost everywhere). If it is everywhere nondegenerate, then we can consider the inverse tensor A_{ij} or the corresponding differential 2-form $\omega = A_{ij}\,dx^i\wedge dx^j$. This form, as is easy to see, is a symplectic structure, i.e., it is nondegenerate and closed.

Definition 4. A function f is called a *Casimir function* of the Poisson structure A if

$$\{f,g\}\equiv 0$$

for any smooth function g.

If the Poisson structure is degenerate, then locally, in a neighborhood of a point in general position, the Casimir functions always exist. Moreover, the number of functionally independent Casimir functions is exactly the corank of the Poisson structure, $\operatorname{corank} A=\dim M-\operatorname{rank} A$.

Example 2. One of the most important examples of Poisson brackets are linear Poisson brackets or Poisson–Lie brackets. Linearity means that the coefficients of the tensor field $A_{ij}(x)$ are linear functions of the coordinates $x_1,\dots,x_n$ (here it is convenient to interchange the superscripts and the subscripts). It is easy to see that there is a natural one-to-one correspondence between such brackets and Lie algebras. Indeed, let G be a finite-dimensional Lie algebra, and let G^* be its dual space. On G^* we define the Poisson bracket by the formula

$$\{f, g\}(x) = x([df(x), dg(x)]), \qquad x \in G^*, \qquad df(x), dg(x) \in (G^*)^* = G.$$

Equivalently, in coordinates, this bracket can be written as

$$\{f, g\}(x) = c^i_{jk} x_i \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k},$$

where the c^i_{jk} are the components of the structural tensor of the algebra G in the basis corresponding to the coordinates $x_1, \dots, x_n$.

Conversely, if we have a linear Poisson bracket, i.e., if $A_{jk}(x) = c^i_{jk} x_i$, then c^i_{jk} is the structural tensor of some Lie algebra.

The Casimir functions of the Poisson–Lie bracket are exactly the invariants of the co-adjoint representation of the corresponding Lie group $\mathfrak{G}$ on G^*.

Now we shall study the situation in which the vector field is Hamiltonian with respect to two compatible Poisson structures. Such vector fields are said to be *bi-Hamiltonian*. It is well known that bi-Hamiltonian vector fields usually have a large supply of first integrals. These integrals can be obtained by using two constructions, which we shall discuss below separately in the degenerate and nondegenerate cases (e.g., see [1, 3, 7, 9, 13]).

1.2. The case of nondegenerate Poisson structures

Suppose that we have two compatible Poisson structures A_0 and A_1 one of which, say A_0, is nondegenerate.

The principal distinction of such a situation from the degenerate case is that the so-called recursion operator $P = A_1 A_0^{-1}$, which plays an important role in our further constructions and is of interest in itself, can be well defined. If A_i is regarded as an operator from the cotangent space $T^*_x M$ into the tangent space $T_x M$, then P is a linear operator in the tangent space, i.e., $P\colon T_x M \to T_x M$. Together with this operator, we shall consider the adjoint operator $P^*\colon T^*_x M \to T^*_x M$, which, obviously, is given by the formula $P^* = A_0^{-1} A_1$.

The following statement shows that the compatibility condition for A_0 and A_1 can be reformulated in terms of the recursion operator.

Proposition 1. *Let A_0 and A_1 be Poisson structures, and let A_0 be nondegenerate. Then the following conditions are equivalent*:

1) *the Poisson structures A_0 and A_1 are compatible*;
2) *the differential 2-form $\Omega_1 = A_0^{-1} P$ determined by the identity $\Omega_1(\xi, \eta) = A_0^{-1}(P\xi, \eta)$ is closed*;
3) *the Nijenhuis tensor corresponding to the recursion operator P is identically zero.*

Here we do not prove this statement. The proof can be found, for instance, in [5, 6, 14]. We only recall that the *Nijenhuis tensor* corresponding to an arbitrary operator P is defined as a bilinear vector-valued form on pairs of vector fields u and v; this form is given by the formula

$$N_P(u, v) = -P^2[u, v] + P[Pu, v] + P[u, Pv] - [Pu, Pv].$$

The following statement shows that the pair of compatible Poisson structures A_0 and A_1 allows us to construct the entire family of pairwise compatible structures A_k $(k \in \mathbb{N})$.

Proposition 2. *A bivector field of the form $A_{k+1} = A_1 (P^*)^k$ is the Poisson structure for any $k \in \mathbb{N}$; moreover, all such structures are pairwise compatible and also compatible with A_0 and A_1.*

Proof. For example, see [13]. We only explain that the expression $A_1(P^*)^k$ stands for a bilinear form on the cotangent space and this form is defined by the natural formula $A_{k+1}(a,b) = A_1({P^*}^k a, b)$. The fact that this form is skew symmetric readily follows from the definition of the recursion operator, i.e., indeed, A_{k+1} is a bivector. □

Remark. In the above situation, it is usually said that the Poisson structure A_k determines a hierarchy of Poisson structures. Nevertheless, it should be noted that all of them can be obtained from the original structures A_0 and A_1 by using the standard tensor operations and hence they are not independent in the natural sense. Moreover, they satisfy a system of linear relations whose coefficients are the coefficients of the characteristic polynomial of the recursion operator. Indeed, if $Q(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$ is the characteristic polynomial for P (or, which is the same, for P^*), then the Hamilton–Cayley theorem implies

$$ {P^*}^n + a_{n-1}{P^*}^{n-1} + \dots + a_1 P^* + a_0 E = 0. $$

Hence, for any $m > n$, we have

$$ A_m + a_{n-1}A_{m-1} + \dots + a_1 A_{m-n+1} + a_0 A_{m-n} = 0. $$

Now we shall study the bi-Hamiltonian vector field

$$ v = \operatorname{sgrad}_0 f_1 = \operatorname{sgrad}_1 f_0. $$

The construction, which allows us to obtain the set of its first integrals and usually ensures complete integrability, consists in the following.

Let us consider a vector field of the form $\operatorname{sgrad}_1 f_1$. It turns out that this field is again bi-Hamiltonian, i.e., there exists a function f_2 such that

$$ \operatorname{sgrad}_1 f_1 = \operatorname{sgrad}_0 f_2. \tag{1} $$

Proceeding in (1) by induction, we obtain the system of recursion relations

$$ \operatorname{sgrad}_1 f_k = \operatorname{sgrad}_0 f_{k+1}. $$

We claim that all the functions f_k are the first integrals of a bi-Hamiltonian field v and that these integrals are mutually commuting.

It is easy to see that, in terms of the recursion operator, these relations can be written in the natural form $df_{k+1} = {P^*}^{k+1}\,df_0$ (i.e., $df_{k+1} = P^*\,df_k$) or as $\operatorname{sgrad}_1 f_k = P^k v$ in terms of Hamiltonian vector fields.

Proposition 3. 1) *Vector fields of the form $P^k v$ are Hamiltonian with respect to the Poisson structures A_0 and A_1. In other words, there exist smooth functions f_k such that* $\operatorname{sgrad}_1 f_k = P^k v$. *These functions satisfy the system of recursion relations*

$$ \operatorname{sgrad}_1 f_k = \operatorname{sgrad}_0 f_{k+1}. $$

2) *The functions f_k pairwise commute with respect to the Poisson brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$. In particular, all of them are first integrals of the bi-Hamiltonian field v.*

3) *If A_1 is nondegenerate, then the vector fields $P^m v$ are Hamiltonian with respect to each of the Poisson structures $A_k = P^k A_0$. The functions f_m pairwise commute with respect to each of the corresponding brackets $\{\ ,\ \}_k$.*

Remark. Both of the Poisson structures A_0 and A_1 are tensor invariants of a bi-Hamiltonian vector field v. This readily implies that the characteristic polynomial $\det(A_1 - \lambda A_0)$ is invariant. In particular, its roots (i.e., the eigenvalues of the recursion operator), considered as functions on a manifold, are first integrals of the vector field v.

There is a natural question: What Hamiltonians yield bi-Hamiltonian vector fields for a given pair of compatible Poisson structures A_0 and A_1? A local answer was obtained by Olver in [12] for different types of A_0 and A_1 We here restrict ourselves to two simplest but important examples.

Example 3. Suppose that the eigenvalues of a recursion operator are real, distinct, and constant. Then there (locally) exists a system of coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ in which

$$\{p_i, p_j\}_0 = \{q_i, q_j\}_0 = 0, \qquad \{p_i, q_j\}_0 = \delta_{ij},$$
$$\{p_i, p_j\}_1 = \{q_i, q_j\}_1 = 0, \qquad \{p_i, q_j\}_1 = a_i\delta_{ij},$$

where $a_1, \dots, a_n$ are the eigenvalues of the recursion operator. It is easy to verify that bi-Hamiltonian systems are determined exactly by Hamiltonians with separated variables:

$$H = F_1(p_1, q_1) + F_2(p_2, q_2) + \dots + F_n(p_n, q_n).$$

The condition under which the hierarchy (1) generates a complete set of first integrals can be written in the following invariant form. Let L_i be two-dimensional eigensubspaces of the recursion operator P. Then $dH(L_i) \neq 0$ for any $i = 1, \dots, n$.

Example 4. Suppose that the eigenvalues of a recursion operator are distinct and all of them are not constant. It turns out that in this case they are automatically functionally independent pairwise commuting functions. In particular, this ensures the complete integrability of any bi-Hamiltonian system in this case. We also note that it is possible to take functions of the form

$$H_k(x) = \operatorname{Tr} P^k(x)$$

to be Hamiltonians yielding nontrivial bi-Hamiltonian systems. It is easy to verify that they are bi-Hamiltonian by using the fact that the Nijenhuis tensor N_P is equal to zero. This relation can be rewritten equivalently as $P\mathcal{L}_v P - \mathcal{L}_{Pv} P = 0$ (here $\mathcal{L}_v$ is the Lie derivative along the vector field v). Then for any vector field v we have (see [13])

$$dH_k(v) = \mathcal{L}_v \operatorname{Tr} P^k = \operatorname{Tr} \mathcal{L}_v P^k = \operatorname{Tr} kP^{k-1}\mathcal{L}_v P = \operatorname{Tr} kP^{k-2}\mathcal{L}_{Pv} P = \frac{k}{k-1} \operatorname{Tr} \mathcal{L}_{Pv} P^{k-1}$$
$$= \frac{k}{k-1}\mathcal{L}_{Pv} \operatorname{Tr} P^{k-1} = \frac{k}{k-1} dH_{k-1}(Pv) = \frac{k}{k-1} P^* dH_{k-1}(v).$$

So we have

$$dH_k = \frac{k}{k-1} P^* dH_{k-1},$$

which means that this field is bi-Hamiltonian. We note that the functions of the form $\operatorname{Tr} P^k$ always yield bi-Hamiltonian systems. However, these functions produce a complete set of integrals in involution only if the spectrum is simple and the eigenvalues are not constant. Under these assumptions, any bi-Hamiltonian system is completely integrable.

The local structure of a nondegenerate pair of compatible Poisson structures is described in more detail in [12, 13].

1.3. The case of degenerate Poisson brackets

Now we consider a linear family of degenerate Poisson structures $J = \{\lambda_0 A_0 + \lambda_1 A_1\}$ on a manifold M (such situation is studied in detail in [1, 9, 15]). For each structure $C \in J$, we can determine its rank, denoted $\operatorname{rank} C$.

Since we assume that here we study only degenerate brackets, we have $\operatorname{rank} C < \dim M$. Let $R_0 = \max_{C \in J} \operatorname{rank} C$. Clearly, almost all brackets from this family have rank R_0. We shall say that they are *brackets in general position*. Only finitely many brackets can be an exception (up to proportionality). The subfamily of brackets in general position will be denoted by J_0.

Let $\mathcal{Z}(C)$ be the set of Casimir functions of a Poisson structure $C \in J$. The following assertion gives a method for constructing a large set of functions in involution with respect to all the brackets from the family under study.

Proposition 4. 1) *Consider two arbitrary brackets $B = \lambda_0 A_0 + \lambda_1 A_1$ and $B' = \mu_0 A_0 + \mu_1 A_1$ from the family J. Suppose that they are not proportional, i.e., $\lambda_0 \mu_1 - \lambda_1 \mu_0 \neq 0$. Then the Casimir functions of these brackets, $f \in \mathcal{Z}(B)$ and $g \in \mathcal{Z}(B')$, are in involution with respect to any bracket $C \in J$.*

2) *The set of Casimir functions $\mathcal{Z}(B)$ is a Lie algebra with respect to any bracket $C \in J$. If $B \in J_0 \subset J$ is a bracket in general position (i.e., if $\operatorname{rank} B = R_0$), then the Lie algebra $\mathcal{Z}(B)$ is commutative.*

Proof. 1) Since the brackets B and B' are not proportional, any bracket $C \in J$ can be represented as the linear combination $C = aB + a'B'$. Then

$$\{f, g\}_C = a\{f, g\}_B + a'\{f, g\}_{B'} = 0,$$

because $f \in \ker\{\ ,\ \}_B = \mathcal{Z}(B)$ and $g \in \ker\{\ ,\ \}_{B'} = \mathcal{Z}(B')$.

2) The fact that $\mathcal{Z}(B)$ is a subalgebra can be easily proved by a straightforward verification. The second statement can be obtained, for example, by passing to the limit as $B' \to B$ in the relation just proved. We note that the condition that B is in "general position" is essential. If this condition does not hold, then, passing to the limit, we obtain, roughly speaking, not all the Casimir functions of the bracket B, but only a subspace. $\square$

So, using a pair of compatible degenerate Poisson brackets, we can construct a sufficiently large family of functions in involution $\mathcal{F}_{J_0}$, combining the Casimir functions of all the brackets in general position,

$$\mathcal{F}_{J_0} = \bigcup_{B \in J_0} \mathcal{Z}(B).$$

Now we assume that we have a dynamical system, which at the same time is a Hamiltonian system with respect to all (nontrivial) brackets from the family under study. Clearly, all the functions from the family $\mathcal{F}_{J_0}$ are its first integrals. Under what conditions is the number of these integrals sufficient for complete integrability? In other words, under what conditions is the number of functionally independent functions in the family equal to

$$\frac{1}{2} \operatorname{rank} C + \operatorname{corank} C = \dim M - \frac{1}{2} R_0,$$

where $C \in J_0$ is a bracket in general position? In fact, this condition means that in $T_x^* M$ the subspace generated by the differentials of functions $g \in \mathcal{F}_{J_0}$ is the maximum isotropic subspace with respect to the form C (almost everywhere on M).

There is an effective criterion [1] for verifying whether the family $\mathcal{F}_{J_0}$ is complete. Moreover, for complete families, this criterion shows at what points of the manifold the first integrals become functionally dependent.

For an arbitrary complex value $\lambda \in \mathbb{C}$, we define the subset of "singular points" in M as follows:

$$S_\lambda = \{x \in M \mid \operatorname{rank}(A_0(x) + \lambda A_1(x)) < R_0\}.$$

Further, we formally set $S_\infty = \{x \in M \mid \operatorname{rank} A_1(x) < R_0\}$.

Now we assume that all the brackets from the family under study have globally defined Casimir functions and these functions locally divide the symplectic leaves of maximum dimension. In other words, if $\operatorname{rank} B(x) = R_0$, then the subspace $\ker B(x)$ is generated by the differentials $df(x)$ of the Casimir functions $f \in \mathcal{Z}(B)$.

Theorem 1 [1]. *Let $L_x \subset T_x^* M$ be the subspace generated by the differentials of all the functions $g \in \mathcal{F}_{J_0}$. Then the following conditions are equivalent*:

1) *L_x is the maximum isotropic subspace with respect to the form $C(x)$, $C \in J_0$*;
2) *$x \notin S = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_\lambda$.*

So the family $\mathcal{F}_{J_0}$ is complete if and only if the complement of the set S is everywhere dense in M. In particular, the condition that all the brackets in the family J have the same rank R_0 is a necessary condition for completeness. Indeed, if this is not the case, then $\operatorname{rank}(A_0 + \lambda A_1) < R_0$ for some $\lambda \in \overline{\mathbb{C}}$ and hence $S_\lambda = M$. However, this condition is not sufficient. Roughly speaking, it is also required that the singular sets S_λ be not too large for each bracket $\{\ ,\ \}_\lambda$.

The following construction allows us to obtain examples of bi-Hamiltonian systems for a given family $\lambda_0 A_0 + \lambda_1 A_1$ of degenerate compatible Poisson brackets. It turns out that one can take the Casimir functions of brackets in general position to be the corresponding Hamiltonians.

Without loss of generality, we assume that A_0 is a bracket in general position and consider the Casimir functions of the linear combination $A_0 - \lambda A_1$ as functions depending on the parameter λ. We assume (and this is just the case in concrete problems) that the dependence on the parameter λ is analytic in a neighborhood of zero (i.e., of the bracket A_0):

$$f_\lambda(x) = f_0(x) + \lambda f_1(x) + \lambda^2 f_2(x) + \cdots \in \mathcal{Z}(A_0 + \lambda A_1).$$

In particular, f_0 is the Casimir function of the bracket A_0.

Proposition 5. *Under the above assumptions, the vector field $v = \operatorname{sgrad}_1 f_0 = A_1(df_0)$ is Hamiltonian with respect to the bracket A_0 and, moreover, with respect to any linear combination $A_0 - \lambda A_1$.*

Proof. Since f_λ is the Casimir function for $A_0 - \lambda A_1$, we have

$$(A_0 - \lambda A_1)(df_0 + \lambda_1 df_1 + \lambda^2 df_2 + \cdots) \equiv 0.$$

Matching the coefficients of like powers of λ, we obtain a system of recursion relations

$$A_0(df_0) = 0, \quad A_0(df_1) = A_1(df_0), \quad \ldots \quad A_0(df_k) = A_1(df_{k-1}), \quad \ldots.$$

The second relation means that the vector field v is Hamiltonian with respect to the bracket A_0 and has the Hamiltonian f_1. Moreover, it is obvious that the vector field v is Hamiltonian with respect to the bracket $A_0 - \lambda A_1$ and has the Hamiltonian $-f_0/\lambda$. The proof of the proposition is complete. $\square$

So, in the case of degenerate Poisson brackets, we have described one of the mechanisms according to which bi-Hamiltonian systems appear. We have also found sufficient conditions for verifying the complete integrability of these systems. In what follows, we show how this construction becomes apparent in special examples in mechanics. However, we point out that the conditions for the family $\mathcal{F}_{J_0}$ to be complete should not be considered as necessary conditions for its integrability. Moreover, there are several methods, allowing us to complete the family $\mathcal{F}_{J_0}$ if it is not complete.

2. THE LAX REPRESENTATION WITH A PARAMETER AND SEMISIMPLE POISSON BRACKETS

Definition 5. A *differential equation in Lax form* is defined to be an equation of the form

$$\frac{dL}{dt} = [L, A],$$

where $L(x)$ and $A(x)$ are some square matrices depending on the phase variables x.

This equation has several remarkable properties. For example, functions of the form $\operatorname{Tr} L^k$ are its first integrals. Sometimes, this equation can be rewritten in the following equivalent form:

$$\frac{dL(\lambda)}{dt} = [L(\lambda), A(\lambda)],$$

where λ is a parameter which is not contained in the equation of motion. In this case the number of integrals increases: for any $\lambda \in \mathbb{R}$, functions of the form $\operatorname{Tr} L(\lambda)$ are also first integrals. Usually, the number of such functions is sufficient for complete integrability.

It is well known that if a Lie algebra G is semisimple, then the equations on G^*, which are Hamiltonian with respect to the Poisson–Lie bracket (see Example 2), can be written in Lax form. Indeed, let us identify the spaces G and G^* by using the inner product $(X, Y) = \operatorname{Tr} XY$, where X and Y are elements of G in some matrix representation. Then on $G^* = G$ the Hamiltonian system with Hamiltonian H can be rewritten as

$$\frac{dX}{dt} = \operatorname{ad}^*_{dH(X)} X, \tag{2}$$

where the operator of the co-adjoint representation ad^* is determined by the identity

$$\langle \operatorname{ad}^*_{dH(X)} Y, \xi\rangle = \langle Y, -[dH(X), \xi]\rangle.$$

Taking into account the above identification, we obtain

$$\operatorname{Tr}(\operatorname{ad}^*_{dH(X)} Y)\xi = -\operatorname{Tr} Y(dH(X)\xi - \xi dH(X)) = \operatorname{Tr}(dH(X)Y - Y dH(X))\xi.$$

Hence we have $\operatorname{ad}^*_{dH(X)} Y = dH(X)Y - Y dH(X) = [dH(X), Y]$. Thus, in the semisimple case, the adjoint and co-adjoint representations coincide and hence the Hamiltonian system (2) can be rewritten as

$$\frac{dX}{dt} = [dH(X), X].$$

Similarly, the following general assertion holds for bi-Hamiltonian systems.

Theorem 2. *Let $\{\ ,\ \}_\lambda$ be a family of Poisson brackets on some linear space. Let almost all of these Poisson brackets be "semisimple" Poisson–Lie brackets (be isomorphic to them). Suppose that a system v is Hamiltonian with respect to all brackets from this family, i.e., it can be represented as*

$$v(x) = \operatorname{sgrad}_\lambda H_\lambda(x),$$

where H_λ is the Hamiltonian corresponding to the bracket $\{\ ,\ \}_\lambda$. Then for v there exists a Lax representation with a parameter λ (which, however, can turn out to be more complicated than a rational one).

Proof. The proof of this fact is rather natural. If the system v is Hamiltonian with respect to the Poisson–Lie bracket corresponding to a semisimple Lie algebra, then, identifying the dual space of the algebra with the algebra itself, we obtain exactly the Lax representation for v (so far without a parameter). Since here we deal with the family $\{\,,\,\}_\lambda$ of semisimple brackets, as the result of the identification (which also depends on λ), we obtain a family of Lax representations with the parameter λ, as was desired. $\square$

Comments. We note that if the family under study contains at least one "semisimple bracket," then almost all its brackets are also semisimple (under the assumption that we deal with linear brackets or brackets reducible to linear ones). Moreover, the entire space of parameters splits into open chambers each of which "contains" mutually isomorphic "semisimple brackets."

Now we demonstrate the above construction by way of several special examples.

3. COMPATIBLE POISSON BRACKETS RELATED TO THE ARGUMENT SHIFT METHOD. EXAMPLES: GEODESICS OF LEFT-INVARIANT METRICS ON SEMISIMPLE LIE GROUPS AND THE TOP ON $so(4)$

Let G be an arbitrary Lie algebra, and let G^* be the dual space. In addition to the standard Poisson–Lie bracket $\{f,g\}(x) = x([df(x), dg(x)])$, on G^* we consider the constant bracket obtained from it by "freezing the argument":

$$\{f,g\}_a(x) = a([df(x), dg(x)]), \qquad \text{where} \quad a \in G^*. \tag{3}$$

It is easy to verify that these brackets are compatible. Therefore, by virtue of the general construction, the Casimir functions of linear combinations of the form $\{\,,\,\} + \lambda\{\,,\,\}_a$ are in involution.

It is easy to see that such a linear combination becomes the standard bracket after the change $x \to x + \lambda a$. In particular, its Casimir functions have the form $f(x + \lambda_a)$, where f runs through the ring of invariants of the co-adjoint representation $I(G)$. So, on the Lie coalgebra, we obtain a set of functions in involution $\{f(x+\lambda a)\}_{f\in I(G),\, \lambda\in\mathbb{R}}$. This method for constructing functions in involution was proposed by Mishchenko and Fomenko [16, 17] as a generalization of Manakov's construction [18]. These authors also showed that such a set of functions is complete in the case of a semisimple Lie algebra. They also described the class of quadratic Hamiltonians for which the functions $f(x+\lambda a)$ are the first integrals. If G is semisimple, then the corresponding systems coincide exactly with systems that are Hamiltonian with respect to any linear combination $\{\,,\,\} + \lambda\{\,,\,\}_a$ (in the case of quadratic Hamiltonians) (see [4]).

Now we describe this construction in more detail. Let G be semisimple. Then, by using the Killing metric, we can identify G with the dual space G^* and assume that the covector a is an element of the algebra G itself. We also assume that a is a semisimple regular element and consider the Cartan subalgebra K generated by this element (in this case it coincides with the centralizer $K = \{x \in G \mid [x,a] = 0\}$ of this element). Now we define a self-adjoint operator $C\colon G \to G$ according to the following rule. We write an arbitrary element $x \in G$ as the sum $x = x_1 + x_2$, where $x_1 \in K$ and $x_2 \in K^\perp$, and write $C(x) = \operatorname{ad}_a^{-1}\operatorname{ad}_b x_2 + D(x_1)$, where $\operatorname{ad}_a^{-1}\colon K^\perp \to K^\perp$ is well defined and $D\colon K \to K$ is an arbitrary self-adjoint operator.

Proposition 6 [16, 17]**.** *The Hamiltonian system*

$$\dot{x} = [Cx, x] \tag{4}$$

with Hamiltonian $h(x) = \frac{1}{2}(Cx, x)$ on the Lie algebra G is completely integrable. Its first integrals are functions of the form $f(x+\lambda a)$, where f is an invariant of the coadjoint representation and

$\lambda \in \mathbb{R}$. *This system is Hamiltonian with respect to any linear combination* $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ *and hence admits the Lax representation with a spectral parameter*:

$$\frac{d}{dt}(x+\lambda a) = [C(x) - b, x + \lambda a]. \tag{5}$$

There is an important special case in which system (4) can be naturally restricted to some subalgebra $H \subset G$ (in other words, the subalgebra H is an "invariant submanifold" of the given system). An example of such a subalgebra is, for instance, provided by the so-called normal form of the complex Lie algebra $G^{\mathbb{C}}$ (see [16, 17]). The Manakov top is the simplest case in this situation. Here $G = sl(n,\mathbb{R})$ and $H = so(n,\mathbb{R}) \subset G$. The subalgebra of diagonal matrices is taken to be the Cartan subalgebra. In particular, a and b are the diagonal matrices $A = \operatorname{diag}(a_1,\dots,a_n)$ and $B = \operatorname{diag}(b_1,\dots,b_n)$. Then the Hamiltonian h, restricted to the subalgebra H, has the form $h = \frac{1}{2}(ad_a^{-1}\operatorname{ad}_b X, X)$, $X \in so(n)$. If we set $A = B^2 = \operatorname{diag}(b_1^2,\dots,b_n^2)$, then $h(X) = \sum(b_i+b_j)^{-1}x_{ij}$ and, substituting this expression into (5), we obtain the Lax representation with spectral parameter presented by Manakov in [18]:

$$\frac{d}{dt}(X+\lambda B^2) = [\Omega - B, X + \lambda B^2],$$

where $\Omega(X) = \mathrm{dh}(X)$ and X are related as $X = B\Omega + \Omega B$.

In [16, 17] Mishchenko and Fomenko showed that such a construction works in the case of arbitrary normal forms. They also described the corresponding integrable systems, which they interpreted as geodesic flows of left-invariant metrics on Lie groups. In particular, on the Lie algebra $so(4)$, they found an interesting system, which can be obtained by repeating the above construction for the pair $H = so(4)$, $G = g_2$. As a result, a new family of quadratic Hamiltonians with an additional integral of power 4 appears on $so(4)$. Later on, this integrable case was rediscovered by Adler and van Moerbeke [19] and by Reyman and Semenov-Tyan-Shansky [20].

The corresponding Hamiltonian system has the following explicit form. We consider a realization of the Lie algebra g_2 as a subalgebra of the Lie algebra $so(4,3)$ generated by matrices of the form

$$\begin{pmatrix} 0 & -\frac{u_3+w_3}{2} & \frac{u_2+w_2}{2} & -\frac{u_1-w_1}{2} & -y_2 & y_3 & a_1 \\ \frac{u_3+w_3}{2} & 0 & -\frac{u_1+w_1}{2} & -\frac{u_2-w_2}{2} & y_1 & a_2 & z_3 \\ -\frac{u_2+w_2}{2} & \frac{u_1+w_1}{2} & 0 & -\frac{u_3-w_3}{2} & a_3 & z_1 & -z_2 \\ \frac{u_1-w_1}{2} & \frac{u_2-w_2}{2} & \frac{u_3-w_3}{2} & 0 & y_3-z_3 & y_2-z_2 & y_1-z_1 \\ -y_2 & y_1 & a_3 & y_3-z_3 & 0 & w_1 & -w_2 \\ y_3 & a_2 & z_1 & y_2-z_2 & -w_1 & 0 & w_3 \\ a_1 & z_3 & -z_2 & y_1-z_1 & w_2 & -w_3 & 0 \end{pmatrix},$$

where $a_1+a_2+a_3 = 0$. The subalgebra $so(4) \subset g_2$ is the maximum compact subalgebra and consists of the matrices corresponding to the variables u_i and w_i. For the Hamiltonian on $so(4) = so(4)^*$, we take a quadratic form of the type

$$H(X) = \operatorname{Tr}(\operatorname{ad}_a^{-1}\operatorname{ad}_b X)X,$$

where a and b are elements of the Cartan subalgebra K consisting of matrices whose entries on the secondary diagonal are arbitrary and the other entries are zero (in this case the element a is assumed to be regular), i.e.,

$$a = \begin{pmatrix} & & & & & & a_1 \\ & & & & & a_2 & \\ & & & & a_3 & & \\ & & & 0 & & & \\ & & a_3 & & & & \\ & a_2 & & & & & \\ a_1 & & & & & & \end{pmatrix}, \qquad b = \begin{pmatrix} & & & & & & b_1 \\ & & & & & b_2 & \\ & & & & b_3 & & \\ & & & 0 & & & \\ & & b_3 & & & & \\ & b_2 & & & & & \\ b_1 & & & & & & \end{pmatrix}.$$

As in the general case, the Lax representation has the form (5), where $x \in so(4) \subset g_2$ and the first integrals have the form $\mathrm{Tr}(X+\lambda a)^k$. In addition to the Hamiltonian and the standard Casimir functions, this yields one more additional integral of degree 4:

$$I_4 = \mathrm{Tr}(2X^4a^2 + 2X^3aXa + X^2aX^2a).$$

After the Casimir functions are excluded, the Hamiltonian and the integral can be represented explicitly as

$$\begin{aligned}
H &= -\frac{2}{3}\sum_k a_i^2a_j^2u_k^2 + \frac{2}{3}\sum_k\left(a_k^4 - a_ia_j\left(a_i^2+a_j^2+\frac{5}{4}a_ia_j\right)\right)w_k^2 \\
&\quad + \sum_k \bigl(a_k^4 + a_i^2a_j^2 - (a_i+a_j)^2(a_i^2+a_j^2)\bigr)u_kw_k, \\
F &= \frac{1}{2}u^2\sum_k\left(\left(a_ia_j - \frac{1}{3}a_k^2\right)w_k^2 + (a_ia_j - a_k^2)u_kw_k\right) \\
&\quad + \frac{1}{18}w^2\sum_k\left(\frac{5}{3}(a_ia_j - a_k^2)w_k^2 + (7a_ia_j - 4a_k^2)u_kw_k\right) \\
&\quad + \frac{1}{2}a^2\left(u^2 + \frac{1}{3}w^2\right)(u,w) - \frac{1}{9}\sum_{i<j}(a_i-a_j)^2(u_iw_iw_j^2 + u_jw_jw_i^2).
\end{aligned}$$

Here $u^2=\sum_k u_k^2$, $w^2=\sum_k w_k^2$, $a^2=\sum_k a_k^2$, and $(u,w)=\sum_k u_kw_k$.

Precisely in the same way, integrable systems can be constructed for other similar pairs $H \subset G$, where H is the normal subalgebra in G. For example, an integrable top can be constructed on the Lie algebra $so(16)$ by using the imbedding $so(16) \subset e(8)$. The integrals obtained in this way will be of power not exceeding 28, while the largest power of the integral for the Manakov top is 14.

4. THE ZHUKOVSKII–VOLTERRA SYSTEM

The construction of the "argument shift" described above admits the following modification in the case of the Lie algebra $so(3)$.

On the Lie algebra $so(3)$ we consider the new Poisson–Lie bracket $\{\,,\,\}_B$ corresponding to the nonstandard commutator $[\,,\,]_B$ on the space of skew-symmetric matrices: $[X,Y]_B = XBY - YBX$, where $B=\mathrm{diag}(b_1,b_2,b_3)$ is a diagonal matrix. In what follows, we discuss such a structure in more detail.

This new bracket is compatible with the standard Poisson–Lie bracket $\{\,,\,\}$ on $so(3)$, as well as with the bracket $\{\,,\,\}_g$ with a "frozen argument," where $g \in so(3)$ is an arbitrary element. We note that the brackets $\{\,,\,\}_B$ and $\{\,,\,\}_g$ are compatible only if $\dim = 3$. For the Lie algebra $so(n)$, this is not the case.

So on the space $so(3)$ we have three pairwise compatible bracket:

$$\{M_i,M_j\} = -\varepsilon_{ijk}M_k, \qquad \{M_i,M_j\}_g = -\varepsilon_{ijk}g_k, \qquad \{M_i,M_j\}_B = -\varepsilon_{ijk}b_kM_k.$$

We consider a family of brackets of the form $\{\,,\,\}_s = s\{\,,\,\} + (\{\,,\,\}_B - \{\,,\,\}_g)$, $s\in\mathbb{R}$. Now we take the Casimir function of the bracket $\{\,,\,\}_B - \{\,,\,\}_g$ to be the Hamiltonian

$$H = \frac{1}{2}(BM,M) - (g,M) = \frac{1}{2}\sum_{i=1}^{3} b_iM_i^3 - \sum_{i=1}^{3} g_iM_i.$$

On the Lie algebra $so(n)$ we obtain the following system, which is Hamiltonian with respect to each bracket of the family under study:

$$\dot{M}_i = \{M_i, dH(M)\}. \tag{6}$$

Indeed, it can be rewritten as $\dot{M}_i = \{M_i, dH_s(M)\}_s$, where the Hamiltonian has the simple form $H_s(M) = \frac{1}{s}H(M)$.

On the other hand, system (6) is the classical Zhukovskii–Volterra system describing the inertial motion of a balanced gyroscope [15, 21].

Because each of the brackets $\{\ ,\ \}_s$ is isomorphic to the standard Poisson–Lie bracket on $so(3)$ under the condition $s + b_i > 0$, we can, following the general construction (Theorem 2), explicitly determine the Lax–Heisenberg representation with spectral parameter for the Zhukovskii–Volterra system. Because the isomorphism between the brackets $\{\ ,\ \}_s$ and $\{\ ,\ \}$ is given by the formula

$$M \to (B + sE)^{-1/2}(M - \widetilde{g})(B + sE)^{-1/2},$$

where

$$M = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}, \qquad \widetilde{g} = \begin{pmatrix} 0 & g_3/(b_3+s) & -g_2/(b_2+s) \\ -g_3/(b_3+s) & 0 & g_1/(b_1+s) \\ g_2/(b_2+s) & -g_1/(b_1+s) & 0 \end{pmatrix},$$

the corresponding L-A-pair has the form (see Theorem 2)

$$L(s) = (B + sE)^{-1/2}(M - \widetilde{g})(B + sE)^{-1/2}, \qquad A(s) = (B + sE)^{1/2} dH(M)(B + sE)^{1/2}.$$

Here

$$dH(M) = \begin{pmatrix} 0 & b_3M_3 - g_3 & -b_2M_2 + g_2 \\ -b_3M_3 + g_3 & 0 & b_1M_1 - g_1 \\ b_2M_2 - g_2 & -b_1M_1 + g_1 & 0 \end{pmatrix}.$$

Somewhat different but equivalent Lax–Heisenberg representations were given in [22] and [23].

5. COMPATIBLE BRACKETS RELATED TO SYMMETRIC PAIRS. THE KOVALEWSKI CASE AND THE RIGID BODY IN A FIELD WITH QUADRATIC POTENTIAL

We consider a semisimple Lie algebra G and its Cartan decomposition $G = H + V$, where H is a subalgebra and V is a subspace satisfying the relations

$$[H, H] \subset H, \qquad [H, V] \subset V, \qquad [V, V] \subset H.$$

This is equivalent to the fact that G admits an involutory diffeomorphism $\theta\colon G \to G$ such that $\theta|_H = \mathrm{id}$ and $\theta|_V = -\mathrm{id}$ (the Cartan involution). In this case the pair (G, H) is called a *symmetric pair*.

The dual space G^* can be represented as $G^* = H^* + V^*$ so that $H^* \perp V$ and $V^* \perp H$.

We consider one more Lie algebra G_θ, which coincides with G as a linear space and whose commutator differs only by the fact that the subspace V is commutative (a commutative ideal): $[V, V] = 0$. For $[H, H]$ and $[H, V]$ the commutator remains the same. Hence, on the dual space G^* there are two different Poisson–Lie brackets $\{\ ,\ \}$ and $\{\ ,\ \}_\theta$ corresponding to the algebras G and G_θ, respectively. It is easy to see that these brackets are compatible.

In addition to these brackets, on G^* there is one more bracket (with frozen argument) $\{\ ,\ \}_a$ described above. If $a \in V^*$, then we can consider any of the commutators $[\ ,\ ]$ and $[\ ,\ ]_\theta$ in formula (3). They will give the same result.

Proposition 7. *If $a \in V^*$, then the brackets $\{\,,\,\}$, $\{\,,\,\}_\theta$, and $\{\,,\,\}_a$ form a family of pairwise compatible Poisson brackets.*

As usual, we assume that G and G^* are identified by using the ad-invariant Killing form. We consider the linear combination $\{\,,\,\}_{\alpha\beta\gamma} = \alpha\{\,,\,\} + \beta\{\,,\,\}_\theta + \gamma\{\,,\,\}_a$. A straightforward calculation shows that the system Hamiltonian with respect to $\{\,,\,\}_{\alpha\beta\gamma}$ and having the Hamiltonian f can be written as

$$\dot h = (\alpha+\beta)([\xi,h]+[\eta,v]) + \gamma[\eta,a], \qquad \dot v = (\alpha+\beta)[\xi,v] + \alpha[\eta,h] + \gamma[\xi,a], \tag{7}$$

where $df(h+v) = \xi + \eta$, $h,\xi \in H$, $v,\eta \in V$.

Proposition 8. *If $\alpha+\beta \neq 0$, then the Poisson bracket $\{\,,\,\}_{\alpha\beta\gamma}$ is equivalent (can be reduced by a linear change) to the semisimple bracket $\{\,,\,\}$ and hence any system Hamiltonian with respect to $\{\,,\,\}_{\alpha\beta\gamma}$ admits the natural Lax–Heisenberg representation.*

To prove this assertion, we explicitly point out the corresponding L-A-pair. Namely, Eqs. (8) can be rewritten as

$$\dot L = [L, A],$$

where

$$L = \sqrt{\frac{\alpha}{\alpha+\beta}}\,h + v + \frac{\gamma}{\alpha+\beta}\,a, \qquad A = -(\alpha+\beta)\left(\xi + \sqrt{\frac{\alpha}{\alpha+\beta}}\,\eta\right).$$

As a corollary, we consider the special case $\gamma=\alpha$, $\beta=1$ and the corresponding family of brackets $\{\,,\,\}_\theta + \alpha(\{\,,\,\} + \{\,,\,\}_a)$. By Theorem 2 and Proposition 8, any system bi-Hamiltonian with respect to this family admits the Lax–Heisenberg representation with a rational spectral parameter, where

$$L = \lambda h + v + \lambda^2 a, \qquad A = (\alpha+1)(\xi + \lambda\eta).$$

In this case $\lambda = \sqrt{\alpha/(\alpha+1)}$ and $df = \xi+\eta$ is the differential of the Hamiltonian of this system with respect to bracket $\{\,,\,\}_\theta + \alpha(\{\,,\,\} + \{\,,\,\}_a)$. It is also necessary that $\alpha \neq -1$.

For the Hamiltonian of the bi-Hamiltonian system in this case, we can consider the Casimir function f of the singular bracket $\{\,,\,\}_\theta - (\{\,,\,\} + \{\,,\,\}_a)$ (this bracket is distinguished in the family by the fact that it is semisimple). In the above case, there always exists a quadratic (inhomogeneous) Casimir function of the form

$$f(h,v) = \frac{1}{2}(C(h),h) + (v,b), \tag{8}$$

where $(\,,\,)$ is the Killing form, $C\colon H \to H$ is a self-adjoint operator, $b \in V$, and, moreover, C and b satisfy the relations $[a,b]=0$ and $[b,h] + [C(h),a] = 0$ (this is exactly equivalent to the fact that f is a Casimir function).

Proposition 9. *Consider the following Hamiltonian system with respect to bracket $\{\,,\,\}_\theta$ on G^*_θ with the Hamiltonian* (9):

$$\dot h = [C(h),h] + [b,v], \qquad \dot v = [C(h),v].$$

This system is bi-Hamiltonian with respect to the family of brackets $\{\,,\,\}_\theta + \alpha(\{\,,\,\} + \{\,,\,\}_a)$ ($\alpha \neq -1$) and hence admits the Lax–Heisenberg representation with a spectral parameter. This representation has the form

$$\dot L = [L, A], \qquad \textit{where} \quad L = \lambda h + v + \lambda^2 a, \quad A = -C(h) - \lambda b.$$

To illustrate the above construction, we consider the Brun system describing the motion of a rigid body in a field with quadratic potential. The integrability of this system was proved by Bogoyavlenskii [24]:

$$\begin{cases} \dot{M} = [M, \omega] - [u, I], \\ \dot{u} = [u, \omega]. \end{cases} \tag{9}$$

In this case $G = gl(3)$, $H = so(3)$, V is the space of symmetric 3×3 matrices, $M \in so(3)$ is the matrix of the angular momentum, $u \in V$ is the matrix whose entries are quadratic with respect to the direction cosines, $\omega = \omega(M) \in so(3)$ is the angular velocity of the rigid body (moreover, $\omega_i = a_i M_i$, where a_i are the inverse of the principal moments of inertia), and $I = \operatorname{diag}(I_1, I_2, I_3)$ is the matrix of inertia of the rigid body.

The Hamiltonian of the system has the form $H = \frac{1}{2}(AM, M) - \operatorname{Tr} uI$ (in the sense of the bracket $\{\,,\,\}_\theta$). As is easy to see, this function is the Casimir function of the bracket $\{\,,\,\}_\theta - \{\,,\,\} - \{\,,\,\}_B$, where B is a symmetric matrix of the form $B = \operatorname{diag}(1/(a_2a_3), 1/(a_3a_1), 1/(a_1a_2))$. According to the general construction, this system is bi-Hamiltonian with respect to the family of brackets $\{\,,\,\}_\theta + \alpha(\{\,,\,\} + \{\,,\,\}_B)$ (where $\alpha \neq -1$) and hence can be written as the Lax pair with a spectral parameter:

$$\frac{d}{dt}(\lambda M + u + \lambda^2 B) = [\lambda M + u + \lambda^2 B, \omega - \lambda I].$$

A similar construction for the pair $G = so(3,2)$ and $H = so(3) \oplus so(2)$ implies the L-A-pair for the Kovalewski top, which was discovered by Reyman and Semenov-Tyan-Shansky, as well as for its generalizations [25, 26].

We obtain this pair using the method presented above. For the algebra G, we take the algebra $so(3,2)$ of 5×5 matrices such that $X^\top = -J_{3,2} X J_{3,2}$, where

$$J = \operatorname{diag}(1,1,1,-1,-1), \qquad X = \begin{pmatrix} \pi_1 & \\ S^\top & \pi_2 \end{pmatrix}, \qquad \pi_1 \in so(3), \qquad \pi_2 \in so(2),$$

and S is a 3×2 matrix. In the Cartan decomposition, the subalgebra H is the direct sum $so(3) \times so(2)$, and V consists of matrices of the form

$$\begin{pmatrix} 0 & S \\ S^\top & 0 \end{pmatrix}.$$

In the variables M, α, and β, the equations of motion of the generalized Kovalewski top in two homogeneous fields are specified by the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - \alpha_1 - \beta_2 \tag{10}$$

and by the Poisson bracket determined by the algebra $so(3) \oplus \mathbb{R}^6$ (it is not necessary to write arbitrary constants x, y before the components α, β in (10), since the structure of this algebra is invariant under the similarity transformations $\alpha \to x\alpha$ and $\beta \to y\beta$, which change only the orbit).

The elements h and v and the shift of the argument I are given by the matrices

$$h = \begin{pmatrix} 0 & M_3 & -M_2 & & \\ -M_3 & 0 & M_1 & \multicolumn{2}{c}{0} \\ M_2 & -M_1 & 0 & & \\ & 0 & & 0 & M_4 \\ & & & -M_4 & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} & & & \alpha_1 & \beta_1 \\ & 0 & & \alpha_2 & \beta_2 \\ & & & \alpha_3 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & & \\ \beta_1 & \beta_2 & \beta_3 & & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} & & & 1 & 0 \\ & 0 & & 0 & 1 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & 0 \end{pmatrix}.$$

In this case we have

$$L = \lambda h + v + \lambda^2 I. \tag{11}$$

We consider a system Hamiltonian with respect to the bracket $\{\ ,\ \}$. The Hamiltonian of this system is determined by the quadratic Casimir function of the bracket $\{\ ,\ \}_\theta - \{\ ,\ \} - \{\ ,\ \}_I$:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2 + M_4^2) - \alpha_1 - \beta_2. \tag{12}$$

This system corresponds to the spherical top on the algebra $so(3,2)$ in a force field. This system is integrable, and its matrix A generated by dH has the form

$$A = \omega_0 - \lambda I, \qquad \text{where} \quad \omega_0 = \begin{pmatrix} 0 & -M_3 & M_2 & & \\ M_3 & 0 & -M_1 & \multicolumn{2}{c}{0} \\ -M_2 & M_1 & 0 & & \\ & 0 & & 0 & -M_4 \\ & & & M_4 & 0 \end{pmatrix}.$$

It is easy to show that equations of motion admit the linear integral

$$M_3 + M_4 = c = \text{const}. \tag{13}$$

Using this integral, we perform a reduction somewhat different from the usual reduction with respect to the momentum. We denote the integrals commuting with the Hamiltonian (12) (in the structure of $\{\ ,\ \}_\theta$) by

$$F_i(M, \alpha, \beta, M_4), \qquad i = 1, 2, 3,$$

and take the new variables M, α, β, $y = cM_3 + M_4$ satisfying the following properties:

1) M, α, and β form a closed subalgebra with respect to the bracket $\{\ ,\ \}_\theta$;
2) y commutes (with respect to $\{\ ,\ \}_\theta$) with the Hamiltonian (12) and the integrals F_i, $i = 1, 2, 3$.

The following assertion holds.

Proposition 10. *The system with Hamiltonian* $\overline{H}(M, \alpha, \beta) = H(M, \alpha, \beta, y)|_{y=c}$, *where* c *is a constant, has involutory first integrals of the form*

$$\overline{F}_i(M, \alpha, \beta) = F_i(M, \alpha, \beta, y)|_{y=c}.$$

Proof. This assertion can be proved by verifying the relations $\{\overline{F}_i, \overline{H}\}_\theta = 0$ and $\{\overline{F}_i, \overline{F}_j\}_\theta = 0$ directly. So the reduced system is also integrable, and its Hamiltonian has the form

$$\overline{H} = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - cM_3 - \alpha_1 - \beta_2.$$

The matrix A corresponding to $d\overline{H}$ has the form

$$A = \omega - \lambda I, \qquad \text{where} \quad \omega = \begin{pmatrix} 0 & -2M_3 + c & M_2 & & \\ 2M_3 - c & 0 & -M_1 & \multicolumn{2}{c}{0} \\ -M_2 & M_1 & 0 & & \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}.$$

Then it is necessary to set $M_4 = c - M_3$ in the matrix L. A complete set of first integrals can be obtained by decomposing $\operatorname{Tr} L^k$ with respect to the spectral parameter.

We note that the above reduction procedure, carried out for the algebra $(so(3) \oplus so(2)) \oplus_S \mathbb{R}^6$ contained in the bundle, cannot be carried out simultaneously for all the brackets in the bundle and, respectively, cannot induce the (reduced) bi-Hamiltonian structure. Apparently, the Kovalewski top (in contrast to the integrable systems studied above) does not admit any bi-Hamiltonian description at all.

We note that a similar construction for the Lie algebra $G = su(2,1)$ allows us to obtain the Lax representation with a spectral parameter for the Goryachev–Chaplygin top. $\square$

6. LIE BUNDLES AND THE MULTIDIMENSIONAL EULER AND CLEBSH CASES

Now we shall consider another topic from the theory of (finite-dimensional) Lie algebras.

Definition 6. A *Lie bundle* on a finite-dimensional space L is defined to be a linear family of Lie structures. This means that on L there is a family of commutators $[\,,\,]_C$ each of which determines the structure of a Lie algebra on L, the parameter C runs through some linear space V, and, moreover, the following natural conditions of linearity are satisfied:

$$[\,,\,]_{\lambda A+\mu B} = \lambda[\,,\,]_A + \mu[\,,\,]_B \qquad \text{for} \quad A, B \in V, \quad \lambda, \mu \in V.$$

This definition is quite similar to the notion of compatible Poisson structures. Indeed, repeating Definition 3, it is natural to say that two Lie structures are *compatible* if any of their linear combinations again determines the structure of a Lie algebra on L (i.e., satisfies the Jacobi identity). This is precisely equivalent to the fact that the structures generate a two-dimensional Lie bundle. Conversely, any Lie bundle consists of pairwise Lie structures.

The relation between the Poisson brackets and the Lie bundles is very simple. If on the linear space L a Lie bundle $[\,,\,]_C$, $C \in V$, is given, then on the dual space L^* a family of pairwise compatible Poisson–Lie brackets $\{\,,\,\}_C$, $C \in V$, arises.

By way of an example, we consider the space of skew-symmetric matrices L, which is identified with the Lie algebra $so(n)$. Introducing the natural invariant inner product $(X, Y) = -\operatorname{Tr} XY$, we identify L with L^*. Next, on L we consider the family of Lie algebras whose commutators are given as

$$[X, Y]_C = XCY - YCX,$$

where C is an arbitrary symmetric matrix. On the dual space $L^* = L$, these algebras generate a family of Poisson–Lie brackets $\{\,,\,\}_C$. The fact that a system v is Hamiltonian with respect to bracket $\{\,,\,\}_C$ means that

$$v(X) = X\,dH(X)C - C\,dH(X)X$$

for some smooth function $H(X)\colon L \to \mathbb{R}$.

We note that

1) all these brackets are mutually compatible,
2) the bracket $\{\,,\,\}_C$ is semisimple if and only if the matrix C is nondegenerate.

In particular, the second property implies that if the matrix C is nondegenerate, then the above equation can be written in Lax form (i.e., as the usual commutator). To this end, we need to make the change

$$X \to C^{1/2}LC^{1/2}, \qquad dH(X) \to C^{-1/2}AC^{-1/2}.$$

After the substitution, we obtain $C^{1/2}\dot{L}C^{1/2} = C^{1/2}(LA - AL)C^{1/2}$ or, which is the same, $\dot{L} = [L, A]$.

Now we consider the Euler equations of motion of a multidimensional rigid body:

$$\dot{X} = X\Omega - \Omega X, \tag{14}$$

where X, Ω are skew-symmetric matrices related as $X = B\Omega + \Omega B$ and B is a diagonal nondegenerate matrix.

In this case one of the Lax representations with a spectral parameter is well known [18]. Here we describe one more representation [2] related to the family of brackets described above. It was shown in [1] that Eqs. (14) are Hamiltonian with respect to the family of brackets $\{\,,\,\}_{B^2+\lambda E}$. Using this assertion and the fact that this bracket is semisimple almost for all λ, we can rewrite equations for each Lie algebra $[\,,\,]_{B^2+\lambda E}$ in Lax form. We present the final result.

Proposition 11. *The system of Eqs.* (14) *can be rewritten in the following equivalent form*:

$$\frac{dL(\lambda)}{dt} = [L(\lambda), A(\lambda)],$$

where

$$L(\lambda) = (B^2+\lambda E)^{-1/2} X (B^2+\lambda E)^{-1/2}, \qquad A(\lambda) = (B^2+\lambda E)^{-1/2}(\lambda\Omega - B\Omega B)(B^2+\lambda E)^{-1/2}.$$

The fact that this representation is equivalent to system (14) can be easily proved by straightforward calculations. However, here the relationship between this fact and the family of brackets is of interest in itself. We shall comment on this relation. Because system (14) is Hamiltonian with respect to the bracket $\{\,,\,\}_{B^2+\lambda E}$, we can represent $\dot X$ as

$$\dot X = X\,dH_\lambda(X)(B^2+\lambda E) - (B^2+\lambda E)\,dH_\lambda(X)X.$$

It is easy to verify that here we have

$$dH_\lambda(X) = (B^2+\lambda E)^{-1}(\lambda\Omega - B\Omega B)(B^2+\lambda E)^{-1}.$$

Now, to obtain a representation with the usual commutator from this relation, we need to make the change already pointed out above

$$X = (B^2+\lambda E)^{1/2} L(\lambda)(B^2+\lambda E)^{1/2}, \qquad dH_\lambda(X) = (B^2+\lambda E)^{-1/2} A(\lambda)(B^2+\lambda E)^{-1/2},$$

which leads to the desired result.

Remark. We note that if the diagonal elements b_i of the matrix B are different, then, for $\lambda = -\min b_i^2$, the Lie algebra $[\,,\,]_{B^2+\lambda E}$ is isomorphic to the Lie algebra $e(n-1) = so(n-1)+\mathbb{R}^{n-1}$ of the group of motions of an $(n-1)$-dimensional (affine) space. Thus we see that, by using a linear change of variables, one can reduce the equations of motion of an n-dimensional rigid body (14) to some integrable system on the algebra $e(n-1)$. It turns out that this system coincides with the $(n-1)$-dimensional analog of the Clebsh case of the motion of a rigid body in an ideal fluid, which was discovered by Perelomov [25]. For $n = 4$, this unexpected isomorphism between two integrable problems was established by Bobenko [27]; the general case is discussed in [2].

7. ANOTHER EXOTIC BUNDLE RELATED TO THE STEKLOV AND LYAPUNOV CASES

Here we discuss another example of the Lie bundle, which Fedorov [28] found analyzing Kötter's classical paper [29].

We consider the direct sum of the spaces of the skew-symmetric matrices $L = so(n) + so(n)$. We write the elements of this space as pairs (X, Y), where $X \in so(n)$ and $Y \in so(n)$. A pair of commutators generating a bundle has the form

$$[(X_1,Y_1),(X_2,Y_2)]_0 = ([X_1,X_2], [X_1,Y_2]+[Y_1,X_2]-[X_1,X_2]_B),$$
$$[(X_1,Y_1),(X_2,Y_2)]_1 = ([X_1,X_2]_B, [Y_1,Y_2]).$$

Here $[\,,\,]_B$ stands for a commutator of the form $[X_1,X_2]_B = X_1BX_2 - X_2BX_1$, where B is a symmetric matrix. In our case we assume that this matrix is diagonal.

It is easy to verify that these commutators are compatible, i.e., any of their linear combinations satisfies the Jacobi identity and hence determines a Lie algebra structure on the space $so(n)+so(n)$.

Now we intend to describe, up to isomorphism, the Lie algebras from the bundle

$$[\,,\,]_{0+\lambda\cdot 1} = [\,,\,]_0 + \lambda[\,,\,]_1.$$

Lemma 1. *If* $\lambda \neq 0$ *and* $\det(E+\lambda B) \neq 0$, *then the Lie algebra* $[\,,\,]_{0+\lambda\cdot 1}$ *is isomorphic to* $so(n)+so(n)$ *with the standard matrix commutator. In this case the isomorphism* φ *is determined by the explicit formulas*

$$\varphi(X,Y) = ((E+\lambda B)^{1/2}X(E+\lambda B)^{1/2}, \lambda Y + X).$$

Proof. This assertion is proved by a straightforward verification. □

This statement readily implies the form of the invariants of the co-adjoint representation on L^*. As usual, we identify $L = so(n)+so(n)$ with $L^* = (so(n)+so(n))^*$ by using the inner product $\langle (X,Y),(Z,P)\rangle = \operatorname{Tr}(XZ+YP)$. The operators $\varphi^*\colon L^* \to L^*$ and $\varphi^{*-1}\colon L^* \to L^*$ have the form

$$\varphi^*(Z,P) = ((E+\lambda B)^{1/2}Z(E+\lambda B)^{1/2} + P, \lambda P),$$
$$\varphi^{*-1}(Z,P) = ((E+\lambda B)^{-1/2}(Z-\lambda^{-1}P(E+\lambda B)^{-1/2}), \lambda^{-1}P).$$

In the standard representation, the invariants of the direct sum $L = so(n)+so(n)$ are well known. These are functions of the form $\operatorname{Tr} Z^{2k}$, $\operatorname{Tr} P^{2k}$. Using (14) and the explicit form of the operator φ^{*-1}, we obtain the explicit form of the Casimir functions of the bracket $\{\,,\,\}_{0+\lambda\cdot 1}$:

$$\operatorname{Tr}((Z-\lambda^{-1}P)(E+\lambda B)^{-1})^{2k}, \qquad \operatorname{Tr} P^{2k}.$$

For $\lambda = 0$, this formula does not work very well. To obtain good asymptotics at zero, we need to consider the following invariant instead of the first invariant:

$$\begin{aligned}&\frac{1}{\lambda}(\operatorname{Tr}((\lambda Z-P)(E+\lambda B)^{-1})^{2k} - \operatorname{Tr} P^{2k})\\ &\qquad = \frac{1}{\lambda}(\operatorname{Tr}((\lambda Z-P)(E-\lambda B+\lambda^2B^2+\lambda^3B^3-\cdots))^{2k} - \operatorname{Tr} P^{2k}).\end{aligned}$$

It is easy to verify that the expression obtained is a series in powers of λ and the first (free) term of this series has the form

$$\operatorname{Tr}(Z+PB)P^{2k-1}.$$

Clearly, this is the Casimir function of the bracket $\{\,,\,\}_0$. Along with functions of the form $\operatorname{Tr} P^{2k}$, they give a complete set.

What is the structure of the Lie algebra $[\,,\,]_0$? It turns out that this structure is isomorphic to the semidirect sum of the algebra $so(n)$ and the commutative ideal $\mathbb{R}^{[n(n-1)/2]}$ with respect to the adjoint representation. The standard commutator for this semidirect sum is determined on the space L in the natural way as

$$[(X_1,Y_1),(X_2,Y_2)]^{\sim} = ([X_1,X_2],[X_1,Y_2]+[Y_1,X_2]).$$

An isomorphism between these standard commutator and the "deformed" commutator $[\,,\,]_0$ is given by the mapping

$$\psi(X,Y) = \left(X, Y - \frac{1}{2}(BX+XB)\right).$$

Recall that this relation means the following:

$$\psi[(X_1,Y_1),(X_2,Y_2)]_0 = [\psi(X_1,Y_1),\psi(X_2,Y_2)]^{\sim}.$$

The adjoint operator has the form

$$\psi^*(Z, P) = \left(Z - \frac{1}{2}(BP + PB), P\right).$$

So the Euler equations in the sense of the bracket $\{\ ,\ \}_0$ can be reduced the standard equations in the sense of the bracket corresponding to the semidirect sum $so(n) +_{\mathrm{ad}} R^{[n(n-1)/2]}$ by using a change of the form $(Z, P) \to (M, P)$,

$$Z = M - \frac{1}{2}(BP + PB), \qquad P = P.$$

Now we describe the family of Hamiltonians generating systems that are Hamiltonian with respect to each bracket of our family. It is easy to see that the Casimir functions of the brackets $\{\ ,\ \}_0 + \lambda\{\ ,\ \}_1$ of maximum rank satisfy this property. Since we are interested only in quadratic Hamiltonians, we can consider a family of functions that are linear combinations of the quadratic Casimir functions described above. It is easy to verify that the functions in this family have the general form

$$H(Z, P) = \sum_{i<j} \frac{c_i - c_j}{b_i - b_j} Z_{ij}^2 + 2\sum_{i<j} \frac{b_i c_i - b_j c_j}{b_i - b_j} Z_{ij} P_{ij} + \sum_{i<j} \frac{b_i^2 c_i - b_j^2 c_j}{b_i - b_j} P_{ij}^2 + \mathrm{const} \sum_{i<j} P_{ij}^2. \quad (15)$$

The last term in this sum is of little importance, because it is the Casimir function for each of the brackets.

Proposition 12. *Suppose that the Hamiltonian H has the form* (15)*. Then it generates a bi-Hamiltonian system. Namely, there exists a function $\widetilde{H}$ such that the identity*

$$\mathrm{sgrad}_1 H = \mathrm{sgrad}_0 \widetilde{H} \quad (16)$$

holds. In this case the Hamiltonian $\widetilde{H}$ can be taken in the form

$$\widetilde{H}(Z, P) = \sum_{i<j} \frac{b_i c_i - b_j c_j}{b_i - b_j} Z_{ij}^2 + 2\sum_{i<j} \frac{b_i^2 c_i - b_j^2 c_j}{b_i - b_j} Z_{ij} P_{ij} + \sum_{i<j} \frac{b_i^3 c_i - b_j^3 c_j}{b_i - b_j} P_{ij}^2.$$

Note that the Hamiltonian $\widetilde{H}$ is not uniquely defined. An arbitrary Casimir function of the bracket $\{\ ,\ \}_0$ can always be added to this Hamiltonian.

Recall that relation (16) can be interpreted as an isomorphism between a system on the semidirect sum $so(n) +_{\mathrm{ad}} \mathbb{R}^{[n(n-1)/2]}$ and a system on the direct sum $so(n) \oplus so(n)$. However, here both brackets are not of exactly standard form; to reduce them to the standard form, some changes are required (they were described above). As a result, we obtain the following result.

Proposition 13. *On the space $G = so(n) \oplus so(n)$ with the standard bracket, consider a Hamiltonian of the form*

$$H_G(X, Y) = \sum_{i<j} \frac{c_i - c_j}{b_i - b_j} b_i b_j X_{ij}^2 + 2\sum_{i<j} \frac{b_i c_i - b_j c_j}{b_i - b_j} \sqrt{b_i b_j} X_{ij} Y_{ij} + \sum_{i<j} \frac{b_i^2 c_i - b_j^2 c_j}{b_i - b_j} Y_{ij}^2.$$

On the space $F = (so(n) +_{\mathrm{ad}} \mathbb{R}^{[n(n-1)/2]})^*$ with the standard Poisson bracket, consider a Hamiltonian of the form

$$H_F(M, P) = \sum_{i<j} \frac{b_i c_i - b_j c_j}{b_i - b_j} \left(M_{ij} - \frac{1}{2}(b_i + b_j) P_{ij}\right)^2 + 2\sum_{i<j} \frac{b_i^2 c_i - b_j^2 c_j}{b_i - b_j} \left(M_{ij} - \frac{1}{2}(b_i + b_j) P_{ij}\right) P_{ij} + \sum_{i<j} \frac{b_i^3 c_i - b_j^3 c_j}{b_i - b_j} P_{ij}^2.$$

Then the systems corresponding to these Hamiltonians can be reduced to each other by the following linear change of variables:

$$M = B^{1/2} X B^{1/2} + \frac{1}{2}(BY + YB), \qquad P = Y.$$

How can we now obtain the Lax representation with a spectral parameter for the Hamiltonians from the family described above?

We consider the bi-Hamiltonian vector field

$$v = \operatorname{sgrad}_1 H = \operatorname{sgrad}_0 \widetilde{H}.$$

As we know, this field can be represented as a Hamiltonian vector field with respect to the linear combination $\{\ ,\ \}_0 + \lambda\{\ ,\ \}_1$:

$$v = \operatorname{sgrad}_{0+\lambda\cdot 1} H_{0+\lambda\cdot 1}. \tag{17}$$

In this case the Hamiltonian has the explicit form

$$H_{0+\lambda\cdot 1} = \sum_{i<j} \frac{a_i - a_j}{b_i - b_j} Z_{ij}^2 + 2\sum_{i<j} \frac{b_i a_i - b_j a_j}{b_i - b_j} Z_{ij} P_{ij} + \sum_{i<j} \frac{b_i^2 a_i - b_j^2 a_j}{b_i - b_j} P_{ij}^2,$$

where $a_i = c_i b_i (1 + \lambda b_i)^{-1}$.

Now we can rewrite Eq. (17) in Lax form, using the above considerations (Theorem 2):

$$\dot{L} = [L, A].$$

In our case we have $L = \varphi^{*-1}(Z, P)$, i.e.,

$$L = \begin{pmatrix} (E + \lambda B)^{-1/2}(Z - \lambda^{-1} P)(E + \lambda B)^{-1/2} & 0 \\ 0 & \lambda^{-1} P \end{pmatrix}, \qquad A = \varphi(dH_{0+\lambda\cdot 1}(Z, P)).$$

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