Geometrical interpretation of Benenti systems

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Abstract

We show that the following two separately developed theories, the theory of Benenti systems in mathematical physics and the theory of projectively equivalent metrics in classical differential geometry, study essentially the same object. Combining methods and results from these two theories, one can prove the commutative integrability of projectively equivalent pseudo-Riemannian metrics and construct infinitely many new Hamiltonian systems, integrable in the classical and in the quantum sense

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1 Introduction

1.1 Benenti systems

Let $g = g_{ij}$ be a Riemannian metric on a smooth manifold M^n of dimension n.

Definition 1. *A nondegenerate (1,1) tensor field L on M^n is called Benenti tensor field with respect to the metric g, if L is self-adjoint and satisfies the following conditions:*

1. *The Nijenhuis tensor of L is identically zero.*
2. *For the functions $H \stackrel{\text{def}}{=} \frac{1}{2} g^{ij} p_i p_j$, $F \stackrel{\text{def}}{=} g^{i\alpha} L^j_\alpha p_i p_j$, we have*

$$\{H, F\} = 2H \cdot \left(\frac{\partial \ \mathrm{trace}(L)}{\partial x^\alpha} g^{\alpha\beta} p_\beta \right),$$

where $\{\ ,\ \}$ denotes the standard Poisson bracket on T^*M^n and x^α, p_β are the standard coordinates on T^*M^n.

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Under these conditions, the geodesic flow of the metric g admits n commuting integrals of a certain form, see [2], [8]. If the eigenvalues of L are all different at least st one point of the manifold, then the integrals are functionally independent almost everywhere on the cotangent bundle to some neighborhood of the point. Below we will show that if the eigenvalues of Benenti tensor field L are all different at one point of the manifold then it is so at almost every point of the manifold; then the integrals are functionally independent almost everywhere and the geodesic flow of g is Liouville-integrable.

1.2 Metrics with the same geodesics

Definition 2. *Two metrics g and $\bar{g}$ on M^n are called* projectively equivalent*, if they have the same geodesics considered as unparameterised curves.*

Metrics with the same geodesics is a very classical subject. They were already studied by Italian mathematical school of the 19th century. In 1865, Beltrami [1] found the first examples of projectively equivalent metrics and showed that any metric, projectively equivalent to the round metric of the sphere, is itself the round metric of the sphere. In 1869, Dini obtained a local description of projectively equivalent metrics on surfaces. In 1896, Levi-Civita found a local description of projectively equivalent metrics on manifolds of arbitrary dimension. Later, metrics with the same geodesics were considered by Weyl, Eisenhart, E. Cartan, Thomas, Lichnerowicz, Venzi, Voss, Pogorelov, Mikes, Aminova, Sinjukov, Solodovnikov. They found a lot of beautiful tensor properties of projectively equivalent metric, see the review paper [19] for details.

However, the global behaviour of projectively equivalent metrics is not understood. Most known global results on projectively equivalent metrics require additional strong geometrical assumptions. For example, for Einstein or (hyper)Kahlerian metrics beautiful results were obtained by Lichnerowicz [12], Venzi [23], Mikes [19] and Hasegawa and Fujimura [7].

An explanation for this is that Dini's and Levi-Civita's theorems describe the metrics in the neighborhood of the points where the eigenvalues of one metric with respect to the other do not bifurcate and it is not always the case: in [17], it is proved that if the manifold is not covered by the torus then projectively equivalent metrics must have points where the eigenvalues bifurcate. The nature of these bifurcations is very important for understanding the global behaviour of projectively equivalent metrics. In Section 4, we show that the eigenvalues behave quite regularly. In particular, if they are all different at some point, the multiplicity of any eigenvalue at any other point can not be greater than three.

2 Main results

Let g be a Riemannian metric on M^n, let $\bar g$ be a (possible, pseudo-Riemannian) metric on M^n. Consider the (1,1)-tensor field L given by the formula

$$L^i_j \overset{\text{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j} \tag{1}$$

Remark 1. *If $\bar g$ is a Riemannian metric then the determinant $\det(\bar g)$ is positive and the tensor field L is well-defined. If n is even, $n+1$ is odd and $(\det(\bar g))^{\frac{1}{n+1}}$ is always well-defined. If n is even and $\det(\bar g)$ is negative, we can consider $-\bar g$ instead of $\bar g$; this changes the sign of $\det(\bar g)$ and makes L well-defined. Later, we will always assume that the sign of $\bar g$ is chosen so that $(\det(\bar g)/\det(g))^{\frac{1}{n+1}}$ is well-defined.*

Theorem 1. *The metrics g and $\bar g$ are projectively equivalent, if and only if L is Benenti tensor field for the metric g.*

As we already mentioned before, one of the most interesting features of metrics admitting Benenti tensor field is the integrability of the geodesic flow, see [2, 3, 8].

Corollary 1 ([13]). *Let the metrics g, $\bar g$ be projectively equivalent. For any parameter t, consider the $(1,1)$- tensor field*

$$S_t \overset{\text{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}. \tag{2}$$

Let us identify the tangent and cotangent bundles of M^n by g. Consider the standard Poisson structure on T^*M^n. Then for any t_1, t_2, the functions

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(\xi) \overset{\text{def}}{=} g(S_{t_i}(\xi), \xi) \tag{3}$$

are commuting integrals for the geodesic flow of g.

Remark 2. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor field S_t, and therefore the function I_t, is well-defined for any t. Moreover, as we will show in Section 4, S_t is a polynomial (in t) of degree $n-1$ with coefficients being (1,1)-tensor fields.*

The theories of Benenti systems and of projectively equivalent metrics are developed separately. Theorem 1 suggests to apply the methods of one theory in the other. We will formulate a few results of such application in Section 5.

The paper is organised as follows. In Section 3, we prove the main theorem. In Section 4, we show that the eigenvalues of L behave quite regularly: they are globally ordered and the points of bifurcation of the eigenvalues are organised into nowhere dense closed subset of measure zero. In particular, if the eigenvalues of L are all different at one point of a connected manifold, then it is so at almost each point of the manifold and the geodesic flow is Liouville-integrable. In Section 5, we discuss projective equivalence of pseudo-Riemannian metrics, Topalov-Sinjukov hierarchy of Benenti systems with potential and quantum integrability.

3 Proof of main theorem

The main goal of this section is to prove Theorem 1. In order to do it, we first formulate necessary and sufficient condition for two metrics to be projectively equivalent.

Theorem 2. *Let g be a Riemannian metric, L a non-degenerate self-adjoint operator. Consider the metric $\bar{g}$ defined by $\bar{g}(\xi,\eta) = \frac{1}{\det L} g(L^{-1}\xi,\eta)$. The metrics g and $\bar{g}$ are projectively equivalent, if and only if the following relation holds for any three vectors u, v and w:*

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v,u) \cdot d\theta(Lw) + \frac{1}{2} g(w,u) \cdot d\theta(Lv). \tag{4}$$

where ∇ is the Riemannian symmetric connexion related to the metric g and $\theta \stackrel{\mathrm{def}}{=} \ln \det L$.

Remark 3. *Since $\nabla_u L$ is a self-adjoint operator, the condition (4) can be equivalently rewritten as*

$$g((\nabla_u L)v, v) = g(v,u) \cdot d\theta(Lv). \tag{5}$$

Proof of Theorem 2: Let $\gamma(t)$ be a geodesic of the metric g parametrised by an arbitrary parameter t. The condition that $\gamma(t)$ is a geodesic can then be formulated as follows: the covariant derivative of the velocity vector $\dot{\gamma}$ along γ is parallel to $\dot{\gamma}$. Analytically this means that $(\nabla_{\dot{\gamma}}\dot{\gamma}) \wedge \dot{\gamma} = 0$.

It is clear that g and $\bar{g}$ have the same geodesics, if and only if

$$(\nabla_{\dot{\gamma}}\dot{\gamma} - \bar{\nabla}_{\dot{\gamma}}\dot{\gamma}) \wedge \dot{\gamma} = 0.$$

Here $\bar{\nabla}$ is the symmetric connexion related to $\bar{g}$.

The difference between ∇ and $\bar{\nabla}$ can be considered as a vector-valued bilinear symmetric form which we denote by $A(u,\xi) \stackrel{\mathrm{def}}{=} \nabla_u\xi - \bar{\nabla}_u\xi$.

Thus, the projective equivalence of g and $\bar{g}$ means that for any ξ

$$A(\xi,\xi) \wedge \xi = 0,$$

or, equivalently, $A(\xi,\xi)$ is parallel to ξ. It is easy to see that this condition is satisfied if and only if

$$A(\xi,\xi) = 2l(\xi)\xi$$

or, since A is symmetric,

$$A(u,\xi) = \nabla_u\xi - \bar{\nabla}_u\xi = l(u)\xi + l(\xi)u.$$

for a certain linear functional l.

The next step is to find the explicit form of the functional l. To this end, let us compute the difference between the divergences corresponding to g and $\bar{g}$. By using the above formula we get immediately

$$\mathrm{div}_g\xi - \mathrm{div}_{\bar{g}}\xi = (n+1)l(\xi).$$

On the other hand, using the standard formula for divergence

$$\mathrm{div}_g \xi = \sum_i \left(\frac{\partial \xi^i}{\partial x^i} + \frac{\partial \ln \sqrt{\det g}}{\partial x^i} \xi^i \right)$$

we see that

$$\mathrm{div}_g \xi - \mathrm{div}_{\bar g} \xi = \frac{1}{2} d \left(\ln \frac{\det g}{\det \bar g} \right) (\xi).$$

Therefore, $l = \frac{1}{2(n+1)} d \left(\ln \frac{\det g}{\det \bar g} \right) = \frac{1}{2} d\theta$. Finally, the metrics g ad $\bar g$ are projectively equivalent, if and only if

$$A(u, \xi) \stackrel{\mathrm{def}}{=} \nabla_u \xi - \bar\nabla_u \xi = \frac{1}{2} \big(d\theta(u)\xi + d\theta(\xi) u \big). \tag{6}$$

The projective equivalence of g and $\bar g$ can be equivalent reformulated in terms of the covariant derivative $\nabla \bar g$: We have

$$\begin{aligned} (\nabla_u \bar g)(\xi, \eta) &= (\nabla_u \bar g)(\xi, \eta) - (\bar\nabla_u \bar g)(\xi, \eta) \\ &= \nabla_u (\bar g(\xi, \eta)) - \bar\nabla_u (\bar g(\xi, \eta)) - \bar g(\nabla_u \xi - \bar\nabla_u \xi, \eta) - \bar g(\xi, \nabla_u \eta - \bar\nabla_u \eta) \\ &= -\bar g(\xi, \eta) \cdot d\theta(u) - \frac{1}{2} \bar g(\xi, u) \cdot d\theta(\eta) - \frac{1}{2} \bar g(\eta, u) \cdot d\theta(\xi), \end{aligned}$$

so that

$$(\nabla_u \bar g)(\xi, \eta) = -\bar g(\xi, \eta) \cdot d\theta(u) - \frac{1}{2} \bar g(\xi, u) \cdot d\theta(\eta) - \frac{1}{2} \bar g(\eta, u) \cdot d\theta(\xi). \tag{7}$$

Here we use the fact that $\nabla_u(\bar g(\xi, \eta)) = \bar\nabla_u(\bar g(\xi, \eta))$ since the covariant derivative of a scalar function coincides with the ordinary derivative along u.

Remark 4. *In the tensor form, conditions (6,7) have been already known to Eisenhart, see [6]. Our proof is nothing else but an invariant reformulation of the original Eisenhart's proof from [6].*

Let us prove that the condition (7) implies the projective equivalence of g and $\bar g$. We need to show that the formula (7) implies

$$A(u, \xi) = \nabla_u \xi - \bar\nabla_u \xi = \frac{1}{2} \big(d\theta(u) \cdot \xi + d\theta(\xi) \cdot u \big). \tag{8}$$

To this end, we note that

$$\nabla_u \bar g(\xi, \eta) = \bar g(A(u, \xi), \eta) + \bar g(A(u, \eta), \xi)$$

Thus, Eisenhart's formula (7) can be considered as a system of linear equations on A. The formula (8) obviously gives a particular solution. Therefore, it is sufficient to show that this solution is unique. Equivalently, this means that the homogeneous system

$$\bar g(A(u, \xi), \eta) + \bar g(A(u, \eta), \xi) = 0$$

admits only trivial solutions.

Using the symmetry of A and this relation, we obtain the following sequence of relations:

$$\begin{aligned}\bar{g}(A(u,\xi),\eta) &= -\bar{g}(A(u,\eta),\xi) = -\bar{g}(A(\eta,u),\xi) = \bar{g}(A(\eta,\xi),u)\\ &= \bar{g}(A(\xi,\eta),u) = -\bar{g}(A(\xi,u),\eta) = -\bar{g}(A(u,\xi),\eta).\end{aligned}$$

Therefore, $\bar{g}(A(u,\xi),\eta) = -\bar{g}(A(u,\xi),\eta) = 0$ or, equivalently, $A \equiv 0$.

Thus, Eisenhart's condition (7) is equivalent to the projective equivalence of g and $\bar{g}$.

Now let us rewrite (7) in terms of the tensor field L. We have

$$\begin{aligned}\nabla_u \bar{g}(\xi,\eta) &= \left(\nabla_u \frac{1}{\det L}\right)\cdot g(L^{-1}\xi,\eta) + \frac{1}{\det L}\cdot g(\nabla_u L^{-1}\xi,\eta)\\ &= -\frac{d(\det L)(u)}{\det L^2}\cdot g(L^{-1}\xi,\eta) - \frac{1}{\det L}\cdot g(L^{-1}(\nabla_u L)L^{-1}\xi,\eta)\\ &= -\frac{1}{\det L}\big(d(\ln\det L)(u)\cdot g(L^{-1}\xi,\eta) + g((\nabla_u L)L^{-1}\xi, L^{-1}\eta)\big).\end{aligned}$$

Substituting this expression in Eisenhart's formula, we get

$$\begin{aligned}&- \frac{1}{\det L}\big(d(\ln\det L)(u)\cdot g(L^{-1}\xi,\eta) + g((\nabla_u L)L^{-1}\xi, L^{-1}\eta)\big) =\\ &- \frac{1}{\det L}\big(g(L^{-1}\xi,\eta)\cdot d\theta(u) + \frac{1}{2}(L^{-1}\xi,u)\cdot d\theta(\eta) + \frac{1}{2}(L^{-1}\eta,u)\cdot d\theta(\xi)\big)\end{aligned}$$

Taking into account that $\theta = \ln\det L$, contracting by $\det L$ and replacing $L^{-1}\xi$ and $L^{-1}\eta$ by v and w, we finally obtain the relation (4). This completes the proof.

Proof of Theorem 1: Let g be a Riemannian metric, L a self-adjoint operator. Consider the metric $\bar{g}$ defined by $\bar{g}(\xi,\eta) = \frac{1}{\det L}g(L^{-1}\xi,\eta)$. Our goal is to prove that the metrics g and $\bar{g}$ are projectively equivalent, if and only if the following two conditions hold:

(i) The Nijenhuis tensor of L vanishes,

(ii) $g(\nabla_u Lu, u) = g(u,u)d$ trace $L(u)$.

It is easy to see that (ii) is equivalent to the second condition in Definition 1.

We first prove that conditions (i), (ii) imply the formula (5) and, consequently, the projective equivalence of the metrics g and $\bar{g}$. We need the following technical statement:

Lemma 1. *If the Nijenhuis tensor of L vanishes then, for any vector u,*

$$d\ \mathrm{trace}L(u) = (d\ \ln\det L)(Lu).$$

Proof: We first notice that for any $(1,1)$-tensor field L and vector field ξ the following identity holds:

$$(d\ln\det L)(\xi) = \mathrm{trace}(L^{-1}\mathcal{L}_\xi L),$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ. Indeed,

$$\begin{aligned}
\mathrm{trace}(L^{-1}\mathcal{L}_\xi L) &= (L^{-1})^i_j\left(\xi^k\frac{\partial L^j_i}{\partial x_k} + L^j_k\frac{\partial \xi^k}{\partial x^i} - L^k_i\frac{\partial \xi^j}{\partial x^k}\right)\\
&= (L^{-1})^i_j\xi^k\frac{\partial L^j_i}{\partial x_k} + \delta^i_k\frac{\partial \xi^k}{\partial x^i} - \delta^k_j\frac{\partial \xi^j}{\partial x^k} = (L^{-1})^i_j\xi^k\frac{\partial L^j_i}{\partial x_k}\\
&= \frac{1}{\det L}\hat{L}^i_j\xi^k\frac{\partial L^j_i}{\partial x_k} = \frac{1}{\det L}\xi^k\frac{\partial \det L}{\partial x_k} = \mathcal{L}_\xi(\ln\det L).
\end{aligned}$$

Here $\hat{L}^i_j$ denotes the element of the co-matrix $\hat{L}$.

Now suppose the Nijenhuis tensor of L vanishes. This means that for any vector field u:

$$\mathcal{L}_{Lu}L = L\mathcal{L}_u L.$$

It follows from this that

$$d\ \mathrm{trace}L(u) = \mathcal{L}_u\mathrm{trace}L = \mathrm{trace}(\mathcal{L}_u L) = \mathrm{trace}(L^{-1}\mathcal{L}_{Lu}L) = (d\ln\det L)(Lu).$$

The lemma is proved.

Therefore, the formula (5) becomes

$$g(\nabla_u Lv, v) = g(v,u)\cdot d\ \mathrm{trace}\ L(v).$$

To prove this identity, we represent $g(\nabla_u Lv, w)$ in the form

$$g(\nabla_u Lv, w) = \frac{1}{2}g(v,u)\cdot d\ \mathrm{trace}\ L(w) + \frac{1}{2}g(w,u)\cdot d\ \mathrm{trace}\ L(v) + B(u,v,w), \quad (9)$$

where $B(u,v,w)$ is a certain tensor. We need to verify that B is actually zero.

Notice that B satisfies one natural symmetry condition

$$B(u,v,w) = B(u,w,v).$$

Besides, condition (ii) immediately implies $B(u,u,u)\equiv 0$. It is easy to see that, together with the symmetry condition, this identity simply means that

$$B(u,v,w) + B(v,w,u) + B(w,u,v) = 0.$$

Finally, let us use the identity $N(L)=0$. It can be equivalently written as

$$(\nabla_{Lu}L)v - L(\nabla_u L)v - (\nabla_{Lv}L)u + L(\nabla_v L)u = 0.$$

From this we get

$$\begin{aligned}
0 &= g((\nabla_{Lu}L)v, w) - g(L(\nabla_u L)v, w) - g((\nabla_{Lv}L)u, w) + g(L(\nabla_v L)u, w)\\
&= g((\nabla_{Lu}L)v, w) - g((\nabla_u L)v, Lw) - g((\nabla_{Lv}L)u, w) + g((\nabla_v L)u, Lw).
\end{aligned}$$

Substituting the representation (9) into this formula, we see that the first two terms of (9) disappear and we obtain one more linear relation on B:

$$B(Lu, v, w) - B(u, v, Lw) - B(Lv, u, w) + B(v, u, Lw) = 0.$$

Thus, we have the following three linear relations on B:

$$B(u, v, w) - B(u, w, v) = 0,$$

$$B(u, v, w) + B(v, w, u) + B(w, u, v) = 0.$$

$$B(Lu, v, w) - B(u, v, Lw) - B(Lv, u, w) + B(v, u, Lw) = 0.$$

Since the metric g is positive definite, all eigenvalues of L are real. Then it is possible to diagonalise L at each point. It turns out that in the case when all the eigenvalues of L are distinct, this system of linear equations admits the zero solution only. More precisely, consider the basis $e_1, \dots, e_n$ such that $L(e_i) = \lambda_i e_i$. Then, setting $u = e_i$, $v = e_j$, $w = e_k$, and using the symmetry with respect to the second and the third arguments of B, we obtain

$$(\lambda_i - \lambda_k)B(e_i, e_j, e_k) + (\lambda_k - \lambda_j)B(e_j, e_k, e_i) = 0.$$

Making the cyclic permutation, we obtain two more relations:

$$(\lambda_j - \lambda_i)B(e_j, e_k, e_i) + (\lambda_i - \lambda_k)B(e_k, e_i, e_j) = 0.$$

$$(\lambda_k - \lambda_j)B(e_k, e_i, e_j) + (\lambda_j - \lambda_i)B(e_i, e_j, e_k) = 0.$$

Together with

$$B(e_i, e_j, e_k) + B(e_j, e_k, e_i) + B(e_k, e_i, e_j) = 0$$

we have a system of 4 homogeneous linear equations which obviously admits only trivial solutions if at least two of the eigenvalues λ_i, λ_j, λ_k are different.

Thus, the only case which needs some additional discussion is $\lambda_i = \lambda_j = \lambda_k = \lambda$. We will use the following general properties of linear operators with zero Nijenhuis tensor.

Proposition 1. *Suppose L is a linear operator on TM^n diagonalisable at each point such that the multiplicities of its eigenvalues are constant in a neighborhood of a point $P \in M$. Suppose that the Nijenhuis tensor of L vanishes. Then the following two statements hold:*

1. *For each eigenvalue λ, the distribution U_λ of the eigenspaces related to λ is integrable.*
2. *Let v be an eigenvector corresponding to λ, and $\mu(x)$ be another eigenvalue considered as a function on M^n. Then the derivative of μ along v is zero.*

Remark 5. *In fact, the following stronger statement takes place. Under the above assumptions, there exist local regular coordinates $x^1, \dots, x^n$ such that in these coordinates $L = diag(\lambda_1, \dots, \lambda_n)$. Moreover, if $\lambda_i \neq \lambda_j$ then $\frac{\partial \lambda_i}{\partial x^j} = 0$.*

This statement is well-known, although we did not find a good reference. We give a prove for self-completeness.

Proof of Proposition 1: 1) Let $u, v \in U_\lambda$ be eigenvectors of L. Then using the standard definition of the Nijenhuis tensor as

$$N_L(u,v) = [Lu, Lv] - L[Lu, v] - L[u, Lv] + L^2[u,v],$$

we obtain the following algebraic relation

$$0 = [\lambda u, \lambda v] - L[\lambda u, v] - L[u, \lambda v] + L^2[u,v] = (L - \lambda E)^2[u,v].$$

Then $[u,v]$ is an eigenvector of L corresponding to the same eigenvalue λ. Thus $[u,v] \in U_\lambda$ and U_λ is, consequently, integrable.

2) Now consider two eigenvectors v, w corresponding to two different eigenvalues λ and μ, respectively. Then substituting v and w into the above formula for the Nijenhuis tensor, we get

$$\begin{aligned} 0 &= [\lambda v, \mu w] - L[\lambda v, w] - L[v, \mu w] + L^2[v,w] \\ &= (L - \lambda E)(L - \mu E)[v,w] + (\lambda - \mu) w(\lambda) v + (\lambda - \mu) v(\mu) w. \end{aligned}$$

Since L is diagonalisable, then neither v nor w (as well as none of its nontrivial linear combinations) belongs to the image of the operator $(L - \lambda E)(L - \mu E)$. Thus, $w(\lambda) = 0$ and $v(\mu) = 0$, as was to be proved.

To complete the proof, we now need one more technical statement. As above, we assume that the multiplicity of λ is locally constant at a point $P \in M$. So we can locally consider the integral submanifold X_λ of the distribution U_λ ($\dim X_\lambda$ is equal to the multiplicity of λ).

Lemma 2. *Assume that the assumptions of Proposition 1 are satisfied and, in addition, L is self-adjoint with respect to a Riemannian metric g on M^n. By $\tilde\nabla$ and $\tilde L$ we denote the Levi-Civita connexion on X_λ and the restriction of L onto TX_λ, respectively. Then the following statements are true:*

1. *For any three vectors $u, v, w \in U_\lambda = T_P X_\lambda$:*

$$g((\nabla_u L)v, w) = g((\tilde\nabla_u \tilde L)v, w).$$

2. *For any $u \in U_\lambda = T_P X_\lambda$:*

$$d\mathrm{trace}\ L(u) = d\mathrm{trace}\ \tilde L(u).$$

Proof: It is well-known that the covariant derivatives ∇ and $\tilde\nabla$ are connected by the following relation:

$$\tilde\nabla_u v = \mathrm{pr}(\nabla_u v),$$

where u, v are vector fields tangent to X_λ and pr is the orthogonal projection onto the tangent space $T_P X_\lambda = U_\lambda$.

It follows from this that for any two vector fields u, v tangent to X_λ:

$$\begin{aligned}\mathrm{pr}((\nabla_u L)v) &= \mathrm{pr}(\nabla_u(Lv) - L\nabla_u v) = \mathrm{pr}(\nabla_u(Lv)) - L(\mathrm{pr}\nabla_u v)\\ &= \tilde\nabla_u(\tilde L v) - \tilde L(\tilde\nabla_u v) = (\tilde\nabla_u \tilde L)v.\end{aligned}$$

Here we use that L is self-adjoint and therefore commutes with the orthogonal projection pr onto its eigenspace U_λ.

Finally, since $g(a, w) = g(\mathrm{pr}(a), w)$ for any $a \in T_pM^n$ and $w \in U_\lambda$, we obtain

$$g((\nabla_u L)v, w) = g(\mathrm{pr}(\nabla_u L)v, w) = g((\tilde\nabla_u \tilde L)v, w),$$

as required.

2) The second statement immediately follows from the second statement of Proposition 1: If u is an eigenvector of L corresponding to λ, then for any other eigenvalue μ of the operator L, we have $d\mu(u) = 0$. This completes the proof.

We now return to the proof of Theorem 1. Let u, v, w be eigenvectors of L related to the same eigenvalue λ. Then, according to the above statement, the relation (7) can be rewritten as follows:

$$g(\tilde\nabla_u \tilde L v, w) = \frac{1}{2} g(v, u) \cdot d\mathrm{trace}\ \tilde L(w) + \frac{1}{2} g(w, u) \cdot d\mathrm{trace}\ \tilde L(v) + B(u, v, w).$$

This means that without loss of generality we can restrict all our considerations to the submanifold X_λ and consider L as a scalar operator of the form $L = \lambda E$.

By putting $v = u, w = u$, we get

$$g(\tilde\nabla_u \tilde L u, u) = g(u, u) \cdot d\mathrm{trace}\ \tilde L(u),$$

or

$$g((\tilde\nabla_u \lambda E)u, u) = g(u, u) \cdot k d\lambda(u) = d\lambda(u) \cdot g(u, u) = g(u, u) \cdot k d\lambda(u),$$

where $k = \dim U_\lambda$. It follows immediately from this that $d\lambda(u) = 0$ for any eigenvector $u \in T_P X_\lambda$ (or $k = 1$, but this case has been already discussed above). In other words, λ is constant on X_λ.

Then for any $u, v, w \in U_\lambda$ the relation (7) becomes trivial

$$0 = 0 + 0 + B(u, v, w).$$

It remains to notice that this is exactly the relation we wanted to prove. This completes the proof of the first part of Theorem 2.

Let us prove the converse statement. We need to show that Eisenhart's formula (7) implies (i), (ii). Let us show first that Eisenhart's formula (7) implies vanishing the Nijenhuis tensor of L.

To do this, we compute $g(N_L(u, v), w)$ for any three vectors $u, v, w \in T_P M^n$ by using (7). We have

$$
\begin{aligned}
g(N_L(u,v),w) &= g((\nabla_{Lu}L)v - L(\nabla_u L)v - (\nabla_{Lv}L)u + L(\nabla_v L)u, w)\\
&= g((\nabla_{Lu}L)v, w) - g((\nabla_u L)v, Lw) - g((\nabla_{Lv}L)u, w) + g((\nabla_v L)u, Lw)\\
&= \frac{1}{2}g(v,Lu)\cdot d\theta(Lw) + \frac{1}{2}g(w,Lu)\cdot d\theta(Lv)\\
&- \frac{1}{2}g(v,u)\cdot d\theta(L^2w) - \frac{1}{2}g(Lw,u)\cdot d\theta(Lv)\\
&- \frac{1}{2}g(u,Lv)\cdot d\theta(Lw) - \frac{1}{2}g(w,Lv)\cdot d\theta(Lu)\\
&+ \frac{1}{2}g(u,v)\cdot d\theta(L^2w) + \frac{1}{2}g(Lw,v)\cdot d\theta(Lu) = 0
\end{aligned}
$$

Thus, the Nijenhuis tensor of L is identically zero.

The relation (ii) follows immediately from formula (5) and Lemma 1.

This completes the proof of the main theorem.

Remark 6. *One can see that we did not use the positive-definiteness of the metric g in proving that projective equivalence of g, $\bar{g}$ implies that L is Benenti tensor field for g. Then this is true also for pseudo-Riemannian metrics. In fact, we used the positive-definiteness of g only once, to prove that the tensor B from (9) is zero.*

4 The eigenvalues of the tensor L

The main goal of this section is to prove the following theorem:

Theorem 3. *Let (M^n, g) be a geodesically complete connected Riemannian manifold. Suppose L is Benenti tensor field for g. At each point $x \in M^n$, denote by*

$$\lambda_1(x) \le \lambda_2(x) \le ... \le \lambda_n(x)$$

the eigenvalues of L at the point x. Then, for any $i \in \{1, ..., n-1\}$, for any $x, y \in M^n$:

1. *$\lambda_i(x) \le \lambda_{i+1}(y)$.*
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$ then $\lambda_i(z) \le \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*
3. *If $\lambda_i(x) = \lambda_{i+1}(y)$ then there exists $z \in M^n$ such that $\lambda_i(z) = \lambda_{i+1}(z)$.*

In a weaker form, Theorem 3 has been announced in [17] and proved in [18].

Corollary 2. *Let (M^n, g) be a geodesically complete connected Riemannian manifold. Suppose L is Benenti tensor field for g. If the eigenvalues of L are all different at one point of M^n then they are all different at almost each point of M^n and the geodesic flow of g is Liouville-integrable.*

For projectively equivalent metrics, this fact has been announced in [14] and proved (by a slightly different method) in [15], see also [18].

Proof of Theorem 3: By definition, the tensor L is self-adjoint with respect to g. Then, for any $x \in M^n$, there exists a basis in T_xM^n such that the metric g is given by the matrix $\operatorname{diag}(1, 1, ..., 1)$ and the tensor L is given by the matrix $\operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \operatorname{diag}(\Pi_1(t), \Pi_2(t), ..., \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Then, for any fixed $\xi = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (3) is the following polynomial in t:

$$I_t = \xi_1^2\Pi_1(t) + \xi_2^2\Pi_2(t) + ... + \xi_n^2\Pi_n(t). \tag{10}$$

Consider the roots of this polynomial. From the proof of Lemma 3, it will be clear that they are real. We denote them by

$$t_1(x, \xi) \le t_2(x, \xi) \le ... \le t_{n-1}(x, \xi).$$

Lemma 3.

1. *For any $\xi \in T_xM^n$,*

   $$\lambda_i(x) \le t_i(x, \xi) \le \lambda_{i+1}(x).$$

   In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$ then $t_i(x, \xi) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. *If $\lambda_i(x) < \lambda_{i+1}(x)$ then for any constant τ the Lebesgue measure of the set*

   $$V_\tau \subset T_xM^n, \quad V_\tau \stackrel{\mathrm{def}}{=} \{\xi \in T_xM^n : t_i(x, \xi) = \tau\},$$

   is zero.

Proof of Lemma 3: Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For any $\alpha \ne i$, we evidently have $\Pi_\alpha(\lambda_i) \equiv 0$. Then

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \Pi_\alpha(\lambda_i)\xi_\alpha^2 = \Pi_i(\lambda_i)\xi_i^2.$$

Hence I_{λ_i} and $I_{\lambda_{i+1}}$ have different signs and therefore the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial I_t. The degree of the polynomial I_t is equal $n-1$; we have $n-1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 3. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then the set

$$V_\tau \stackrel{\mathrm{def}}{=} \{\xi \in T_x M^n : t_i(x,\xi) = \tau\},$$

consists of the points ξ where the function $I_\tau(x,\xi) \stackrel{\mathrm{def}}{=} (I_t(x,\xi))_{|t=\tau}$ is zero; then it is a nontrivial quadric in $T_x M^n \equiv R^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then any coefficient $\Pi_\alpha(t)$ of the quadratic form (10) has a factor $(\lambda_i - t)^{k-1}$. Therefore,

$$\tilde{I}_t \stackrel{\mathrm{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in t and $\tilde{I}_\tau$ is a nontrivial quadratic form. Evidently, for any point $\xi \in V_\tau$, we have $\tilde{I}_\tau(\xi) = 0$ so that the set V_τ is a subset of a nontrivial quadric in $T_x M^n$ and its measure is zero. Lemma 3 is proved.

The first statement of Theorem 3 follows immediately from the first statement of Lemma 3: Let us join the points $x, y \in M^n$ by a geodesic $\gamma : R \to M^n$, $\gamma(0) = x$, $\gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x,\xi)$ and the roots

$$t_1(x,\xi) \le t_2(x,\xi) \le ... \le t_{n-1}(x,\xi).$$

By Corollary 1, each root t_i is constant on each orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)).$$

Using Lemma 3, we obtain

$$\lambda_i(\gamma(0)) \le t_i(\gamma(0), \dot\gamma(0)), \quad \text{and} \quad t_i(\gamma(1), \dot\gamma(1)) \le \lambda_{i+1}(\gamma(1)).$$

Therefore $\lambda_i(\gamma(0)) \le \lambda_{i+1}(\gamma(1))$ and the first statement of Theorem 3 is proved.

Let us prove the second statement of Theorem 3. There exists a sufficiently small neighborhood $U(\gamma(1))$ of the point $\gamma(1)$ such that the point $\gamma(0)$ can be joined with any point of $U(\gamma(1))$ by a geodesic lying in a small tubular neighborhood of the geodesic γ. We assume that any two points of the neighborhood $U(\gamma(1))$ can be joined by a geodesic; for example we can assume that U is a small ball of radius less than the radius of injectivity. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for any point y of some subset $V \subset U(\gamma(1))$. Then by the first statement of Theorem 3, the value of λ_i is a constant (independent of $y \in V$). Indeed, joining any two points $y_0, y_1 \in V$ by a geodesic, we have

$$\lambda_i(y_0) \le \lambda_{i+1}(y_1) \;\text{ and }\; \lambda_i(y_1) \le \lambda_{i+1}(y_0).$$

Denote this constant by C. Let us prove that $\lambda_i(\gamma(0)) = \lambda_{i+1}(\gamma(0)) = C$. Let us join the point $\gamma(0)$ with every point of V by all possible geodesics. Consider the set $V_C \subset T_{\gamma(0)}M^n$ of the initial velocity vectors (at the point $\gamma(0)$) of these geodesics.

By the first statement of Lemma 3, for any geodesic γ_1 passing through any point of V, the value $t_i(\gamma_1, \dot\gamma_1)$ is equal to C. Then, by the second statement of Lemma 3, the measure of the set V_C is zero and therefore the measure of the set V is also zero. The second statement of Theorem 3 is proved.

Let us prove the third statement of Theorem 3. Let $\lambda_i(\gamma(0)) = \lambda_{i+1}(\gamma(1)) = \lambda$ for some $i \in \{1, ..., n-1\}$ and for some constant λ. We will assume that $\lambda_i(\gamma(0)) < \lambda_{i+1}(\gamma(0))$. Let us show that the geodesic γ consists of the points where either λ_i or λ_{i+1} (or both λ_i and λ_{i+1}) are equal to λ.

If t_i is a multiple root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$, or if there exists a point $z \in M^n$ such that $\lambda_{i-1}(z) = \lambda$ then the statement obviously follows from Lemma 3 and the first statement of Theorem 3. Suppose t_i is not a multiple root and $\lambda_{i-1}(z) < \lambda$ for any $z \in M^n$.

Consider the function $I_\lambda : TM^n \to R$. Suppose the differential dI_λ is zero at some point $(z, \nu) \in TM^n$, $\nu \neq 0$. Let us show that then either $\lambda_i(z)$ or $\lambda_{i+1}(z)$ (or both $\lambda_i(z)$ and $\lambda_{i+1}(z)$) are equal to λ.

Indeed, consider the coordinate system such that the metric g at the point z is given by the diagonal matrix $\mathrm{diag}(1, 1, .., 1)$ and the mapping L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, .., \lambda_n)$. Then the restriction of the function I_λ to the tangent space T_zM^n is given by

$$\sum_{\alpha=1}^{n} \xi_\alpha^2 \Pi_\alpha(\lambda)$$

The partial derivatives $\frac{\partial I_\lambda}{\partial \xi_\alpha}$ are

$$\frac{\partial I_\lambda}{\partial \xi_\alpha} = 2\Pi_\alpha(\lambda)\xi_\alpha.$$

Then λ is equal to one of the numbers $\lambda_1, ..., \lambda_n$; by assumption it can be equal to either $\lambda_i(z)$ or $\lambda_{i+1}(z)$.

Now let us show that the differential dI_λ vanishes at every point $(\gamma(\tau), \dot\gamma(\tau))$. Evidently the differential of any integral is preserved by the geodesic flow so that it is sufficient to prove that the differential vanishes at the point $(\gamma(0), \dot\gamma(0))$.

By Lemma 3, we have

$$\lambda = \lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)) \leq \lambda_{i+1}(\gamma(1)) = \lambda$$

so that λ is a root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ and $I_\lambda(\gamma(0), \dot\gamma(0)) = 0$. By assumptions, the eigenvalue λ_i has multiplicity one in a small neighborhood of $\gamma(0)$. Then it is a smooth function on this neighborhood, and the function $I_{\lambda_i(z)}(z, \nu)$ is also smooth on the tangent bundle to this neighborhood. Consider the function $I_\lambda(z, \nu) - I_{\lambda_i(z)}(z, \nu)$. Its differential vanishes at the point $(\gamma(0), \dot\gamma(0))$. More precisely, by assumptions, λ is a simple root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ so that in a neighborhood of the point $\big(\lambda, (\gamma(0), \dot\gamma(0))\big) \in R \times TM^n$ the function $I_t(z, \nu)$ is a monotone function in t. But λ_i is no greater than λ.

Then the difference $I_\lambda(z,\nu) - I_{\lambda_i(z)}(z,\nu)$ is either always non-positive or always non-negative in a small neighborhood of $(\gamma(0),\dot\gamma(0))$. By assumptions, $\lambda_i(\gamma(0)) = \lambda$ so that $I_\lambda(\gamma(0),\dot\gamma(0)) - I_{\lambda_i(\gamma(0))}(\gamma(0),\dot\gamma(0))$ is zero and therefore the function $I_\lambda(z,\nu) - I_{\lambda_i(z)}(z,\nu)$ has a local extremum at the point $(\gamma(0),\dot\gamma(0))$ and its differential vanishes at this point.

Now, the differential of the function $I_{\lambda_i(z)}(z,\nu)$ also vanishes at the point $(\gamma(0),\dot\gamma(0))$. More precisely, as we have shown in the proof of Lemma 3, the function I_{λ_i} is either always non-positive or always non-negative. But $I_{\lambda_i(\gamma(0))}(\gamma(0),\dot\gamma(0))$ is equal to $I_\lambda(\gamma(0),\dot\gamma(0))$ and is zero. Then the point $(\gamma(0),\dot\gamma(0))$ is an extremum of the function I_{λ_i} and therefore the differential of I_{λ_i} vanishes at the point $(\gamma(0),\dot\gamma(0))$.

Thus, the differential of the function I_λ is zero at the point $(\gamma(0),\dot\gamma(0))$. Then it vanishes at each point of the curve $(\gamma(\tau),\dot\gamma(\tau))$. Then at any point τ either $\lambda_i(\gamma(\tau)) = \lambda$ or $\lambda_{i+1}(\gamma(\tau)) = \lambda$. Therefore, any point of the segment $[0,1]$ lies in one of the following sets:

$$\begin{aligned}\Gamma_0 &= \{\tau \in [0,1] : \lambda_i(\gamma(\tau)) = \lambda\},\\ \Gamma_1 &= \{\tau \in [0,1] : \lambda_{i+1}(\gamma(\tau)) = \lambda\}.\end{aligned}$$

The subsets Γ_0, Γ_1 are evidently closed and non-empty. Then they intersect; at each point τ of the intersection we have $\lambda_i(\gamma(\tau)) = \lambda_{i+1}(\gamma(\tau)) = \lambda$. Theorem 3 is proved.

5 Pseudo-Riemannian metrics, Topalov-Sinjukov hierarchy for systems with potential and quantum integrability

In this Section we formulate the first results of combining the theory of Benenti systems and of projectively equivalent metrics. The formal proofs are rather lengthy and will appear elsewhere.

5.1 Commutative integrability for projectively equivalent pseudo-Riemannian metrics

Corollary 1 can be generalised for pseudo-Riemannian metrics:

Theorem 4. *Let pseudo-Riemannian metrics g, $\bar g$ be projectively equivalent. As in Section 2, consider the tensor field L given by (1). For any parameter t, consider the tensor field S_t given by (2). Let us identify the tangent and the cotangent bundles of M^n by g. Consider the standard Poisson structure on T^*M^n. Then, for any t_1, t_2, the functions*

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(\xi) \stackrel{\text{def}}{=} g(S_{t_i}(\xi),\xi)$$

are commuting integrals for the geodesic flow of g.

Sketch of the proof: The fact that the functions I_t are integrals for the geodesic flow of g follows from the construction of integrals for projectively equivalent Riemannian metric given in [21]. One can note that this construction does not use the positive-definiteness of the metrics.

The problem was to prove the commutativity of these integrals; the original proof in [21] essentially uses the local description obtained in [11] for simultaneously-diagonalisable metrics only.

To show that the integrals I_t commute, one can use the bihamiltonian formalism for Benenti systems developed in [8]. One can check that the results of [8] are also true for pseudo-Riemannian metrics so that if L is Benenti tensor for g then the functions I_t commute. Now, by Remark 6, if two pseudo-Riemannian metrics g and $\bar{g}$ are projectively equivalent, the tensor field L is Benenti tensor field for the metric g so that the integrals I_t commute,

5.2 Topalov-Sinjukov hierarchy for Benenti systems with potential

In [20], Sinjukov suggested a construction that, given a pair of projectively equivalent metrics, produces another pair of projectively equivalent metrics. Here we formulate this construction in terms of Benenti tensor field:

Theorem 5 ([20]). *Let g be a Riemannian metric. Suppose the tensor field L is self-adjoint and positive-definite. Denote by g_L the Riemannian metric given by*

$$g_{i\alpha}L^{\alpha}_j.$$

(In invariant terms, the metric g_L is given by $g_L(\xi,\nu) = g(L\xi,\nu)$.) Then L is Benenti tensor field for g_L if and only if it is Benenti tensor field for g.

In other words, given a metric admitting Benenti tensor field L, we can immediately construct another metric g_L such that L is Benenti tensor field also for it. We can apply the construction once more; it is natural to denote the resulting metric by g_{L^2} since it is given by $g_{L^2}(\xi,\nu) = g(L^2\xi,\nu)$. We can go in other direction and construct the metrics $g_{L^{-1}}$, $g_{L^{-2}}$ and so on: L is Benenti tensor field also for each of these metrics.

The oldest example of projectively equivalent metrics on the sphere is due to Beltrami [1]. In view of Theorem 1, this example gives us nontrivial Benenti tensor field for the metric of the round sphere. Strangely enough, Sinjukov did not apply his construction to the example of Beltrami. It has been done in [15], see also [22]. It has been shown that, consequently applying Sinjukov's construction to Beltrami's example, one can get the metric of ellipsoid (as the metric g_L) and the metric of the Poisson sphere (as the metric $g_{L^{-2}}$).

In [2], Benenti formulated in invariant terms when it is possible to separate the variables in the Hamiltonian system with the Hamiltonian being the sum of kinetic and potential energy. Here we recall the part of this theory related to metrics admitting Benenti tensor fields:

Theorem 6 ([2]). *Let g be a Riemannian metric on M^n, let L be Benenti tensor field for g. Let us identify the tangent and the cotangent bundles of M^n by g. Consider the standard Poisson structure on T^*M^n. Consider the tensor fields S_t given by the formula (2). Suppose there exist a function V on M^n and a one-parametric family of functions V_t on M^n such that for any t*

$$S_t(dV) = dV_t. \tag{11}$$

Then, for any t_1, t_2, the functions

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(\xi) \stackrel{\text{def}}{=} g(S_{t_i}(\xi), \xi) + V_{t_i} \tag{12}$$

are commuting integrals for the geodesic flows of the Hamiltonian system with the Hamiltonian $H = g(\xi, \xi) + V$.

It is easy to see that the integrals (12) are functionally independent if and only if the integrals (3) of the geodesic flow of g are functionally independent; in view of Corollary 2, this happens if and only if there exists a point of the manifold where the spectrum of L is simple.

Maybe the most famous example of such a situation is the so-called Neumann system: the motion on the round sphere in the quadratic potential. In this case, the kinetic energy of the system is given by the metric of the round sphere, which, in view of Theorem 1 and Beltrami's examples, admits a non-trivial Benenti tensor field. It can be verified directly that the potential of the Neumann system satisfies condition (11), see [2]. Slightly less known example is a certain generalisation of Neumann system described in [4], see also [2].

We see that condition (11) on the potential V does not depend on the metric g. Therefore, if L is Benenti tensor field for g with simple spectrum at least at one point, and if there exist functions V, V_t satisfying (11) then the Hamiltonian system with the Hamiltonian $H_{L^k} \stackrel{\text{def}}{=} g_{L^k}(\xi, \xi) + V$ is also integrable. Thus, given a Hamiltonian system with the Hamiltonian satisfying the hypotheses of Theorem 6, we can construct a hierarchy of Hamiltonian systems; if the integrals (12) for the initial system are functionally independent, then any system from the hierarchy is Liouville-integrable. Thus, starting from the system of Neumann, one can construct infinitely many integrable Hamiltonian systems, whose kinetic energies include the kinetic energy of the motion on the ellipsoid and the kinetic energy of the motion on the Poisson sphere.

5.3 Quantum integrability for Benenti systems with potential

In [14], it has been shown that the "quantum" version of Corollary 1 is also true (we formulate this theorem in terms of Benenti tensor field):

Theorem 7 ([14]). *If L is Benenti tensor field for the Riemannian metric g then, for any t_1, t_2, the operators $\mathcal{I}_{t_i} : C^2(M^n) \to C^0(M^n)$,*

$$\mathcal{I}_{t_i}(f) \stackrel{\text{def}}{=} \text{div}(S_{t_i}(\text{grad}(f))), \tag{13}$$

where tensor fields S_t are given by (2), commute and commute with the Laplacian of the metric g.

It appears that it is possible to include the potential from Theorem 6:

Theorem 8. *Let g be a Riemannian metric on M^n, let L be Benenti tensor field for g. Consider the tensor fields S_t given by the formula (2). Suppose there exist a function V on M^n and a one-parametric family of functions V_t on M^n satisfying condition (11). Then, for any t_1, t_2, the operators*

$$\mathcal{I}_{t_i} : C^2(M^n) \to C^0(M^n), \quad \mathcal{I}_{t_i}(f) \stackrel{\text{def}}{=} \operatorname{div}(S_{t_i}(\operatorname{grad}(f)) + V_{t_i},$$

commute and commute with the Laplacian of the metric g.

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