

Splitting and gluing constructions for geodesically equivalent pseudo-Riemannian metrics

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Abstract

Two metrics g and \bar{g} are geodesically equivalent, if they share the same (unparameterized) geodesics. We introduce two constructions that allow one to reduce many natural problems related to geodesically equivalent metrics, such as the classification of local normal forms and the Lie problem (the description of projective vector fields), to the case when the $(1, 1)$ -tensor $G_j^i := g^{ik} \bar{g}_{kj}$ has one real eigenvalue, or two complex conjugate eigenvalues, and give first applications. As a part of the proof of the main result, we generalize Topalov-Sinjukov (hierarchy) Theorem for pseudo-Riemannian metrics

1 Introduction

Definition 1. Let g and \bar{g} be Riemannian or pseudo-Riemannian metrics on the same manifold M^n . We say that they are *geodesically equivalent* (notation: $g \sim \bar{g}$), if they have the same geodesics considered as unparameterized curves.

Given two metrics, we consider the $(1, 1)$ -tensor $L = L(g, \bar{g})$ defined by

$$L_j^i := \left(\frac{\det(\bar{g})}{\det(g)} \right)^{\frac{1}{n+1}} \bar{g}^{ik} g_{kj}. \quad (1)$$

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(Replacing g by $-g$ if necessary, we can always assume that (1) is well-defined.)

The goal of this paper is to give two constructions: gluing and splitting.

- The simplified version of the *gluing construction* does the following. Consider two manifolds M_1 and M_2 with pairs of geodesically equivalent metrics $h_1 \sim \bar{h}_1$ on M_1 and $h_2 \sim \bar{h}_2$ on M_2 . Assume that the corresponding $(1, 1)$ -tensor fields $L_1 = L(h_1, \bar{h}_1)$ and $L_2 = L(h_2, \bar{h}_2)$ have no common eigenvalues in the sense that for any two points $x_1 \in M_1$, $x_2 \in M_2$ we have

$$\text{Spectrum } L_1(x_1) \cap \text{Spectrum } L_2(x_2) = \emptyset.$$

Then one can naturally construct a pair of geodesically equivalent metrics $g \sim \bar{g}$ on the direct product $M = M_1 \times M_2$. These new metrics g and \bar{g} differ from the direct product metrics $h_1 + h_2$ and $\bar{h}_1 + \bar{h}_2$ on $M_1 \times M_2$, but can be obtained from them by explicit formulas involving L_1 and L_2 (see (8), (9) below). We denote this by

$$(M, g, \bar{g}) = (M_1, h_1, \bar{h}_1) \times_{(L_1, L_2)} (M_2, h_2, \bar{h}_2)$$

The corresponding $(1, 1)$ -tensor $L = L(g, \bar{g})$ is, however, the direct sum of L_1 and L_2 in the natural sense: for every

$$\xi = \left(\underbrace{\xi_1}_{\in T_{x_1}M_1}, \underbrace{\xi_2}_{\in T_{x_2}M_2} \right) \in T_{(x_1, x_2)}(M_1 \times M_2) \quad \text{we have} \quad L(\xi) = (L_1(\xi_1), L_2(\xi_2)).$$

- The *splitting construction* is the inverse operation. Its local version can be described as follows.

Consider a manifold M with two geodesically equivalent metrics $g \sim \bar{g}$. Assume that at a point $p \in M$ the corresponding $(1, 1)$ -tensor $L = L(g, \bar{g})$ has at least two distinct eigenvalues that are not conjugates of each other. Then choose a partition of the spectrum of L into two disjoint nonempty subsets $S_1 \sqcup S_2$ such that each pair of complex-conjugate eigenvalues lies in the same subset. Equivalently, one can say that the characteristic polynomial $\chi(t)$ of L is factorised into two real polynomials $\chi_1(t) \cdot \chi_2(t)$ with no common roots and S_i is just the set of roots of $\chi_i(t)$.

Then there is a neighborhood $U = U(p) \subset M$ such that in the above notation the triple (U, g, \bar{g}) can be presented as

$$(U, g, \bar{g}) = (U_1, h_1, \bar{h}_1) \times_{(L_1, L_2)} (U_2, h_2, \bar{h}_2),$$

with $\dim U_1 = \deg \chi_1$, $\dim U_2 = \deg \chi_2$. The splitting construction gives an explicit formula for the pair of geodesically equivalent metrics $h_i \sim \bar{h}_i$ on U_i in terms of g, \bar{g}, L and the chosen factorisation $\chi(t) = \chi_1(t) \cdot \chi_2(t)$ (see (5), (6) below).

The splitting construction is, of course, more important: it allows one to “decompose” (M, g, \bar{g}) at least locally into simpler pieces (U_i, h_i, \bar{h}_i) of the same kind. From the topological viewpoint this decomposition (i.e., the existence of two transversal foliations of complementary dimensions on M) is, in fact, very natural and is induced by the $(1, 1)$ -tensor field L itself. We show that the Nijenhuis torsion of L vanishes (Theorem 1) and, as result, the partition $S_1 \sqcup S_2$ of the spectrum of L leads immediately to two natural integrable distributions that define on M the desired foliations (Theorem 2). The geometric part of the construction is much deeper. We show in particular that M carries two hidden locally product pseudo-Riemannian metrics h and \bar{h} which can be canonically reconstructed from g, \bar{g} by means of some non-trivial (and, in our opinion, beautiful) formulas (Theorem 3) involving the information about the partition of the spectrum. Notice that we formulate all of our results in invariant terms which makes it possible to apply them to study global properties of manifolds with geodesically equivalent metrics.

In the Riemannian case, these constructions were obtained in [Ma1, Ma4], and played an important role in recent developments in the theory of geodesically equivalent Riemannian metrics. Roughly speaking, these constructions allow one to reduce many natural questions about geodesically equivalent metrics to the case when the $(1, 1)$ -tensor $L(g, \bar{g})$ has one real eigenvalue, or two complex-conjugate eigenvalues.

We expect similar applications in the theory of pseudo-Riemannian metrics. The main idea is still the same: by iterating the splitting construction, each pair of geodesically equivalent metrics (g, \bar{g}) can be canonically “decomposed” into natural “components” (h_i, \bar{h}_i) , $i = 1, \dots, s$, of smaller dimensions with the same property $h_i \sim \bar{h}_i$, but with a much simpler structure of the tensor $L_i = L(h_i, \bar{h}_i)$. Moreover, some other important geometrical objects appearing in this context inherit this decomposition. Thus, we

are able to work with individual “components” and, as a result, to reduce essentially technical difficulties. We give more details and comments in Sections 2.1, 2.2.

It is worth noticing that compared to Riemannian geometry, the gluing/splitting construction in the pseudo-Riemannian case seems to be even more important because it replaces to some extent the classical Levi-Civita theorem providing a canonical form for a pair of geodesically equivalent Riemannian metrics. The absence of a reasonable analogue of the Levi-Civita theorem in the pseudo-Riemannian case is, in fact, the main reason why the “Riemannian” proof of the gluing/splitting theorems cannot be generalised to the case of pseudo-Riemannian metrics, where one has to develop essentially different techniques (see Remark 6).

A part of our proof of Theorems 3 and 4 could be consider as a separate result. It is a generalisation of a construction by Sinjukov [Si1] and Topalov [To] which makes it possible to build, starting from a given pair of geodesically equivalent metrics $g \sim \bar{g}$, a whole family of geodesically equivalent pairs $g_f \sim \bar{g}_f$. We give an invariant explanation of this construction which works both in the Riemannian and pseudo-Riemannian context and show that such a family is, in fact, very large: as its parameter, f , one can take an arbitrary real-analytic function satisfying certain natural assumptions, see Section 1.3 for more details.

1.1 Splitting construction

Definition 2. A *local-product structure* on M^n is a triple (h, B_1, B_2) , where h is a pseudo-Riemannian metric, and B_1, B_2 are foliations of dimensions r and $n-r$ ($1 \leq r < n$) such that in a neighborhood of each point $p \in M^n$ one can choose local coordinates

$$(x, y) = ((x^1, x^2, \dots, x^r), (y^{r+1}, y^{r+2}, \dots, y^n)),$$

satisfying the following conditions:

- 1) the leaves of B_1 are given by $y = \text{const} \in \mathbb{R}^{n-r}$;
- 2) the leaves of B_2 are given by $x = \text{const} \in \mathbb{R}^r$;

3) the metric h takes the form

$$ds_h^2 = \sum_{i,j=1}^r h_{ij}(x) dx^i dx^j + \sum_{\alpha,\beta=r+1}^n h_{\alpha\beta}(y) dy^\alpha dy^\beta;$$

in other words, the matrix of h is block-diagonal and its first $r \times r$ block depends on the x -coordinates and the second $(n-r) \times (n-r)$ block depends on the y -coordinates:

$$h = \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(y) \end{pmatrix}.$$

Example 1. A model example of manifolds with local-product structure is obviously the direct product of two pseudo-Riemannian manifolds (M_1^r, h_1) and (M_2^{n-r}, h_2) . In this case, the leaves of the foliations B_1 and B_2 are respectively $M_1^r \times \{y\}$, $y \in M_2^{n-r}$ and $M_2^{n-r} \times \{x\}$, $x \in M_1^r$. The metric h is the usual product metric $h_1 + h_2$. Locally, every local-product structure is as in this model example.

Geometrically, the existence of a local-product structure for a given metric h is equivalent to the following condition: $T_p M$ splits into the direct sum of two nontrivial orthogonal subspaces $U, V \subset T_p M$ invariant with respect to the holonomy group. Indeed, if (h, B_1, B_2) is a local-product structure, then $U = T_p B_1$ and $V = T_p B_2$ are such subspaces. If the orthogonal subspaces $U, V \subset T_p M$ (such that $U \oplus V = T_p M$) are invariant with respect to the holonomy group, then, as it was shown by De Rham [DR] and Wu [Wu], the parallel translation of these subspaces does not depend on the curve, and generates integrable distributions, whose integral manifolds form the foliations B_1 and B_2 satisfying Definition 2.

For two metrics g and \bar{g} , we consider the $(1, 1)$ -tensor $L = L(g, \bar{g})$ given by (1). Its characteristic polynomial will be denoted by

$$\chi : M \times \mathbb{R} \rightarrow \mathbb{R}, \quad \chi(t) := \det(t \cdot \text{Id} - L). \quad (2)$$

It is a polynomial of degree n whose coefficients are smooth functions on the manifold. Note that by our definition the polynomial is monic, i.e., its leading coefficient is 1.

We say that a factorisation of $\chi(t)$ into two monic real polynomials

$$\chi(t) = \chi_1(t) \cdot \chi_2(t), \quad \deg \chi_i \geq 1, \quad \chi_i : M \times \mathbb{R} \rightarrow \mathbb{R}$$

is *admissible* if $\chi_1(t)$ and $\chi_2(t)$ are coprime (i.e., have no common roots) at every point $x \in M$.

Example 2. Let $\lambda_1, \dots, \lambda_n$ be (possibly, complex) eigenvalues of L at a point $p \in M$ counted with their algebraic multiplicities. Divide them into two nonempty groups (without loss of generality we think that $\lambda_1, \dots, \lambda_r$ lie in the first group and $\lambda_{r+1}, \dots, \lambda_n$ in the second group) in such a way that $\lambda_i \neq \lambda_\alpha$ for $i = 1, \dots, r$, $\alpha = r + 1, \dots, n$, and pairs of complex-conjugate eigenvalues lie in the same group. Then, the polynomials

$$\chi_1(t) := (t - \lambda_1) \dots (t - \lambda_r), \quad \chi_2(t) := (t - \lambda_{r+1}) \dots (t - \lambda_n)$$

are real and give an admissible factorisation of $\chi(t)$ at $p \in M$. By using the implicit function theorem, it is easy to see that this factorisation can always be extended onto some neighborhood $U(p) \in M$.

Locally every admissible factorisation is as in this example. However, the existence of a local admissible factorisation at every point does not imply its global existence.

Given an admissible factorisation $\chi = \chi_1 \chi_2$, we consider the following two distributions D_1, D_2 on M :

$$D_i = \ker \chi_i(L), \quad i = 1, 2, \tag{3}$$

where \ker denotes the kernel. Here (and in all other places of the paper where we consider polynomials in L with coefficients being smooth functions on M) we treat the $(1, 1)$ -tensor L as a linear operator acting on each tangent space, and a polynomial $f(L)$ in L is the $(1, 1)$ -tensor of the form $f(L) = a_0(x) \cdot \text{Id} + a_1(x)L + a_2(x)L^2 + \dots + a_m(x)L^m$.

It is easy to see that D_1 and D_2 are transversal distributions of complementary dimensions. Moreover, the distributions are invariant with respect to L , and are mutually orthogonal with respect to the both metrics g and \bar{g} .

Theorem 1. *Let g and \bar{g} be geodesically equivalent pseudo-Riemannian metrics on M , $L = L(g, \bar{g})$ be the $(1, 1)$ -tensor associated with them, and $\chi = \chi_1 \cdot \chi_2$ be an admissible factorisation of its characteristic polynomial. Then, the distributions (3) are integrable.*

By Theorem 1, the admissible factorisation implies the (local) existence of a coordinate system $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ such that the distribution D_1 is generated by the coordinate vector fields ∂_{x^i} , $i = 1, \dots, r$, and similarly D_2 is generated by ∂_{y^j} , $j = r+1, \dots, n$. Since the distributions are invariant with respect to L , in this coordinate system the matrix of L is block-diagonal:

$$L(x, y) = \begin{pmatrix} L_1(x, y) & 0 \\ 0 & L_2(x, y) \end{pmatrix},$$

where L_1 and L_2 are $r \times r$ - and $(n - r) \times (n - r)$ - matrices respectively.

Theorem 2. *Under the assumptions of Theorem 1 and in notation above, the entries of L_1 depend on the x -variables only, and the entries of L_2 depend on the y -variables only, so*

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}. \quad (4)$$

By Theorem 1, the distributions D_1 and D_2 generate two foliations on the manifold. We denote them by B_1 and B_2 , respectively.

Theorem 3 (Splitting construction). *Let g and \bar{g} be geodesically equivalent pseudo-Riemannian metrics on M , and B_1, B_2 be the foliations generated by the distributions (3) related to an admissible factorisation $\chi = \chi_1 \cdot \chi_2$ of the characteristic polynomial of $L = L(g, \bar{g})$. Then, the following two $(0, 2)$ -tensors*

$$h_{ij} = g_{ik} \left((\chi_2(L) + \chi_1(L))^{-1} \right)_j^k \quad (5)$$

$$\bar{h}_{ij} = \bar{g}_{ik} \left(\left(\frac{1}{\chi_2(0)} \chi_2(L) + \frac{1}{\chi_1(0)} \chi_1(L) \right)^{-1} \right)_j^k \quad (6)$$

are pseudo-Riemannian metrics (i.e., symmetric and nondegenerate), and the triples (h, B_1, B_2) and (\bar{h}, B_1, B_2) are local-product structures on M . Moreover, for every leaf M_i of the foliation B_i , the restrictions $h_i = h|_{M_i}$ and $\bar{h}_i = \bar{h}|_{M_i}$ of the metrics h and \bar{h} to this leaf are geodesically equivalent, and the tensor $L_i = L(h_i, \bar{h}_i)$ defined on M_i by (1) coincides with the restriction $L|_{M_i}$ of L to M_i , $i = 1, 2$. Moreover, $\chi_1 = \chi_{L_1}$ and $\chi_2 = \chi_{L_2}$.

In formula (6), the expression $\chi_i(0)$, $i = 1, 2$, denotes the smooth function on M obtained by substituting $t = 0$ into $\chi_i(t)$, namely, $\chi_1(0) = (-1)^r \det L_1$, and $\chi_2(0) = (-1)^{n-r} \det L_2$.

Remark 1. It might be convenient to understand the formulas (5, 6) in matrix notation: let us consider the coordinate system $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ such that the distribution D_1 is generated by the vectors ∂_{x^i} , and the distribution D_2 is generated by the vectors ∂_{y^j} . Then, in this coordinate system all matrices we use to construct h and \bar{h} are block-diagonal with one $r \times r$ - and one $(n - r) \times (n - r)$ -block:

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} \bar{g}_1 & 0 \\ 0 & \bar{g}_2 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

Hence, the matrices of h and \bar{h} must be block-diagonal as well, and direct calculations give us the formula

$$h = \begin{pmatrix} g_1 \chi_2(L_1)^{-1} & 0 \\ 0 & g_2 \chi_1(L_2)^{-1} \end{pmatrix}, \quad \bar{h} = \begin{pmatrix} \chi_2(0) \bar{g}_1 \chi_2(L_1)^{-1} & 0 \\ 0 & \chi_1(0) \bar{g}_2 \chi_1(L_2)^{-1} \end{pmatrix} \quad (7)$$

Remark 2. Although the geodesic equivalence relation is obviously symmetric, the tensor L is not invariant with respect to the permutation of g and \bar{g} . More precisely, the tensor (1) constructed for \bar{g} , g is the inverse of (1) constructed for g , \bar{g} . Moreover, two local product structures constructed from the pair of geodesically equivalent metrics g, \bar{g} does not coincide with the local product structure constructed from the pair \bar{g}, g , though of course they are closely related, in particular (after the proper choice of admissible factorisations) all these four local-product structures share the same foliations B_1 and B_2 .

1.2 Gluing construction

Let B_1 and B_2 be two transversal foliations of complementary dimensions, and h, \bar{h} be two metrics such that the triples (h, B_1, B_2) and (\bar{h}, B_1, B_2) are local-product structures.

The tangent spaces to the leaves of the foliations will be denoted by TB_1, TB_2 , and for every $p \in M$ we have $T_p M = T_p B_1 \oplus T_p B_2$.

We denote by h_i, \bar{h}_i the restrictions of the metrics h, \bar{h} to the leaves of B_i , $i = 1, 2$. The corresponding tensor (1) will be denoted by $L_i := L(h_i, \bar{h}_i)$, and its characteristic polynomial by χ_i , $i = 1, 2$. Assume in addition that the polynomials χ_1 and χ_2 are coprime for all points of M .

Now, consider the following two symmetric bilinear forms g, \bar{g} on M : for two tangent vectors

$$u = \left(\underbrace{u_1}_{\in TB_1}, \underbrace{u_2}_{\in TB_2} \right), \quad v = \left(\underbrace{v_1}_{\in TB_1}, \underbrace{v_2}_{\in TB_2} \right) \in TM$$

we put

$$g(u, v) = h_1(\chi_2(L_1)(u_1), v_1) + h_2(\chi_1(L_2)(u_2), v_2) \quad (8)$$

$$\bar{g}(u, v) = \frac{1}{\chi_2(0)} \bar{h}_1(\chi_2(L_1)(u_1), v_1) + \frac{1}{\chi_1(0)} \bar{h}_2(\chi_1(L_2)(u_2), v_2). \quad (9)$$

Theorem 4 (Gluing construction). *If h_1 is geodesically equivalent to \bar{h}_1 , and h_2 is geodesically equivalent to \bar{h}_2 , then the metrics g, \bar{g} given by (8, 9) are geodesically equivalent too.*

Remark 3. It is an easy linear algebra to see that the tensor (1) constructed for the metrics (8, 9) is the direct sum of the tensors L_1 and L_2 : for every $u = (u_1, u_2) \in T_p B_1 \oplus T_p B_2 = T_p M$ we have

$$L(u) = (L_1(u_1), L_2(u_2)). \quad (10)$$

Remark 4. It might again be convenient to understand the formulas (8, 9) in matrix notation: we consider the coordinate system $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ such that y -coordinates are constant on the leaves of B_1 and x -coordinates are constant on the leaves of B_2 . Then, in this coordinate system, the matrices of g and \bar{g} are given by

$$g = \begin{pmatrix} h_1 \chi_2(L_1) & 0 \\ 0 & h_2 \chi_1(L_2) \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} \frac{1}{\chi_2(0)} \bar{h}_1 \chi_2(L_1) & 0 \\ 0 & \frac{1}{\chi_1(0)} \bar{h}_2 \chi_1(L_2) \end{pmatrix} \quad (11)$$

Remark 5. Comparing formulas (7) and (11), we see that the gluing construction is inverse to the splitting.

Remark 6. If the case of Riemannian metrics, Theorems 3,4 were proven in [Ma1, Ma4]. The proof is based on the Levi-Civita description of geodesically equivalent metrics, which is not complete in the pseudo-Riemannian case, see the discussion in Section 2.1. Below, we will show that the Levi-Civita description is an easy corollary of Theorems 3,4, see example in Section 2.1.

1.3 Functions of $(1, 1)$ -tensors and Topalov-Sinjukov Theorem for pseudo-Riemannian metrics

Suppose a compact set $K \subseteq \mathbb{C}$, a function $f : K \rightarrow \mathbb{C}$, and a $(1, 1)$ -tensor L (on M) satisfy the following assumptions:

- (i) $\mathbb{C} \setminus K$ is connected (we do not require that K is connected).
- (ii) $f : K \rightarrow \mathbb{C}$ is a continuous function, and the restriction $f|_{\text{Int } K}$ is holomorphic.
- (iii) K is symmetric with respect to the x -axes: for every $z \in K$ its conjugate \bar{z} also lies in K .
- (iv) for every $z \in K$, $f(z) = \bar{f(\bar{z})}$, where the bar “ $\bar{}$ ” denotes the complex conjugation.
- (v) Spectrum $L(x) \subset \text{Int } K$ for every $x \in M$.

Under the above assumptions (i–v), one can naturally define a $(1, 1)$ -tensor $f(L)$: by Mergelyan’s theorem [Ca], the function f can be uniformly approximated by real polynomials p_i . We define $f(L) = \lim_{i \rightarrow \infty} p_i(L)$. It is an easy exercise (see for example [Hi, §1.2.2 – 1.2.4]) to show that the limit exists, is independent on the choice of the sequence p_i , smoothly depends on $x \in M^m$ (actually, the function $f(L)$ is analytic in the entries of L), and behaves as a $(1, 1)$ -tensor when we change the variables.

Example 3. Polynomials $p(z)$ with real coefficients, the functions e^z , $\cos(z)$, $\sin(z)$ satisfy the above assumptions for every L , and we can naturally consider the standard operator functions $p(L)$, e^L , $\sin(L)$, $\cos(L)$ as smooth $(1, 1)$ -tensor fields on M .

Example 4. Let $\chi = \chi_1 \cdot \chi_2$ be an admissible factorization of the characteristic polynomial of L in a sufficiently small neighborhood $U(x_0) \subset M$ and D_1, D_2 be the corresponding distributions. Then, the natural projectors $P_i : TU \rightarrow D_i$ are functions of L in the above sense. Indeed, consider the corresponding partition $S_1 \sqcup S_2$ of Spectrum $L(x_0)$ into two disjoint subsets and let K_1 and K_2 be ε -neighborhoods of S_1 and S_2 respectively (ε is small enough so that K_1 and K_2 do not intersect). Then, the function

$$f_1 : K \rightarrow \mathbb{C}, \quad f_1(z) := \begin{cases} 1 & \text{for } z \in K_1 \\ 0 & \text{for } z \in K_2. \end{cases}$$

satisfies the above assumptions, and we can define the function $f_1(L)$ which obviously coincides with the projector P_1 . Similarly, one can define $f_2(L) = P_2$.

Notice that for every $x \in M$ we have: Spectrum $f(L(x)) = f(\text{Spectrum } L(x))$. In particular, if for every $x \in M$ we have $0 \notin f(\text{Spectrum } L(x))$, then the tensor $f(L)$ is nondegenerate. If in addition L is g -self-adjoint, then $f(L)$ is also g -self-adjoint, so $g_f := (g_{i\alpha} f(L)_{j\alpha}^i)$ is a metric.

Theorem 5. *Let g and \bar{g} be geodesically equivalent metrics and L be the $(1,1)$ -tensor given by (1). Suppose $K \subset \mathbb{C}$ is a compact set, and $f : K \rightarrow \mathbb{C}$ is a function such that K , f , and L satisfy the assumptions (i – v) above. Assume in addition that for every $x \in M$ we have $0 \notin f(\text{Spectrum } L(x))$.*

Then, the metrics $g_f := (g_{i\alpha} f(L)_j^\alpha)$ and $\bar{g}_f := (\bar{g}_{i\alpha} f(L)_j^\alpha)$ are also geodesically equivalent.

Remark 7. Partial cases of this theorem were proved by Sinjukov [Si1] and Topalov [To]. More precisely, Sinjukov proved this theorem assuming that the function $f(z) = z$. Topalov proved the theorem assuming the metric g is Riemannian (in this case the eigenvalues of L are real, and is sufficient to require that the function f is real-analytic).

2 History, motivation and possible applications

The theory of geodesically equivalent metrics has a long and fascinating history. First non-trivial examples were discovered by Lagrange [La]. Geodesically equivalent metrics were studied by Beltrami [Be], Levi-Civita [LC], Painlevé [Pa] and other classics. One can find more historical details in the surveys [Am2, Mi2] and in the introductions to the papers [Ma1, Ma2, Ma4].

The success of general relativity made necessary to study geodesically equivalent pseudo-Riemannian metrics. The textbooks [Ei, Pe1, Pe2, Hal3] on pseudo-Riemannian metrics have chapters on geodesically equivalent metrics. In the popular paper [We2], Weyl stated few interesting open problems on geodesic equivalence of pseudo-Riemannian metrics. Recent references (on the connection between geodesically equivalent metrics and general relativity) include Hall et al [HL1, HL2, HL3], Hall [Hal1, Hal2], Kiosak et al [KM1], Gibbons et al [GWW].

In the recent time, a huge progress was made in the theory of geodesically equivalent Riemannian metrics. The splitting/gluing constructions played a crucial role in this progress (as we mentioned in Remark 6, in the Riemannian case, Theorems 3, 4 were known). We expect similar applications in the pseudo-Riemannian situation as well. The list of problems where the splitting/gluing constructions were used in the Riemannian case, and are expected to be used in the pseudo-Riemannian case is below; we discuss it in detail in Sections 2.1, 2.2.

Beltrami Problem¹. *Describe all pairs of geodesically equivalent metrics.*

Lie Problem². *Find all metrics g admitting infinitesimal projective transformations.*

Lichnerowicz Conjecture³. *Let a connected Lie group G act on a complete connected manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformations, or for some $c \in \mathbb{R} \setminus \{0\}$ the metric $c \cdot g$ is the Riemannian metric of constant positive sectional curvature $+1$.*

Recall that a *projective transformation* of a Riemannian manifold is a diffeomorphism of the manifold that takes unparameterized geodesics to geodesics. Local projective transformations obviously form a Lie pseudo-group, its generators, i.e., vector fields whose local flow takes unparameterized geodesics to geodesics are called by Lie *infinitesimal projective transformations*. In the modern terminology, they are called *projective vector fields*; we will use this terminology in the paper.

2.1 Motivation I. Normal form of geodesically equivalent metrics: Beltrami problem

If the eigenvalues of L do not bifurcate at a point (this condition is fulfilled almost everywhere), the answer to Beltrami's question was given by Levi-Civita [LC] under the assumption that the metrics are Riemannian.

A local description of geodesically equivalent pseudo-Riemannian metrics, which might be treated as a pseudo-Riemannian analog of the Levi-Civita theorem, is considered to be done by Aminova [Am1]. Unfortunately, the authors of the present paper do not understand her result (we do not doubt that the result is correct). Moreover, we have checked that in all 6 papers which refer to [Am1] according to MathSciNet, the authors

¹ Italian original from [Be]: *La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda*

² German original from [Lie], Abschn. I, Nr. 4, *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten*

³the attribution to Lichnerowicz is folkloric; we did not find a paper of Lichnerowicz where he states this conjecture. Certain papers refer to this statement as to a classical conjecture, see the discussion in [Ma4]. Lichnerowicz conjecture answers the question asked by Schouten in [Sc]

cited Aminova's theorem to give an overview of the subject only, but did not really use it.

The statement of the main theorem of [Am1] is on two pages, and one more page is devoted to explanation of the formulas in the theorem. This is probably the reason why this result is hardly applicable and was, to the best of our knowledge, never used.

The splitting and gluing constructions suppose to make the description of geodesically equivalent pseudo-Riemannian metrics much simpler. Indeed, at almost every point $p \in M$ the eigenvalues of L do not bifurcate, i.e., the algebraic multiplicity of each eigenvalue λ_i is locally constant and is equal to k_i . In a small neighborhood of such a point, the eigenvalues (both real and complex) can be treated as smooth functions of x , and we can factorise the characteristic polynomial of L into a product $\chi = \chi_1 \cdots \chi_m$ of polynomials $\chi_i : \mathbb{R} \times M \rightarrow \mathbb{R}$ of two kinds: either $\chi_i(t) = (t - \lambda_i)^{k_i}$ for a real eigenvalue λ_i , or $\chi_i(t) = (t - \lambda_i)^{k_i} (t - \bar{\lambda}_i)^{k_i}$ for a pair of complex conjugate eigenvalues $\lambda_i, \bar{\lambda}_i$.

Repeatedly applying the splitting construction $m - 1$ times in a small neighborhood $U(p)$, we can construct local coordinates x_1, \dots, x_m , where $x_i = (x_i^1, \dots, x_i^{l_i}) \in \mathbb{R}^{l_i}$ ($l_i = k_i$ if λ_i is real, and $l_i = 2k_i$ for a couple of complex conjugate eigenvalues $\lambda_i, \bar{\lambda}_i$), and pairs of geodesically equivalent metrics $h_i(x_i) \sim \bar{h}_i(x_i)$ such that

- 1) each eigenvalue λ_i of L depends on x_i only: $\lambda_i = \lambda_i(x_i)$,
- 2) the characteristic polynomial of $(1, 1)$ -tensor $L_i = L(h_i, \bar{h}_i)$ associated with h_i and \bar{h}_i by means of (1) is exactly $\chi_i(t)$,
- 3) in this coordinate system, g and \bar{g} simultaneously take the following block diagonal form

$$g = \begin{pmatrix} h_1 \hat{\chi}_1(L_1) & & & \\ & \ddots & & \\ & & h_m \hat{\chi}_m(L_m) & \\ & & & \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} \frac{1}{\hat{\chi}_1(0)} \bar{h}_1 \hat{\chi}_1(L_1) & & & \\ & \ddots & & \\ & & \frac{1}{\hat{\chi}_m(0)} \bar{h}_m \hat{\chi}_m(L_m) & \\ & & & \end{pmatrix} \quad (12)$$

where $\hat{\chi}_i(t) = \prod_{j \neq i} \chi_j(t)$.

Thus we have

Proposition 1. *In a neighborhood of a regular point $p \in M$, the geodesically equivalent metrics g and \bar{g} can be simultaneously reduced by an appropriate choice of local coordinates x_1, \dots, x_m , $x_i \in \mathbb{R}^{l_i}$, to the form (12), where h_i and \bar{h}_i are geodesically equivalent*

metrics depending on x_i -coordinates only and such that the corresponding $(1, 1)$ -tensor $L_i = L(h_i, \bar{h}_i)$ has either one single real eigenvalue $\lambda_i(x_i)$, or a single pair of complex conjugate eigenvalues $\lambda_i(x_i), \bar{\lambda}_i(x_i)$.

Thus, in order to describe geodesically equivalent metrics near a generic point it is sufficient to do this under the assumption that L has one real eigenvalue, or two complex-conjugate eigenvalues. In the Riemannian case, we will illustrate this idea by the following

Example: Levi-Civita Theorem follows from the splitting construction. Let the geodesically equivalent metrics $g \sim \bar{g}$ be Riemannian. Then all eigenvalues of L are real and positive, and L is semi-simple. Hence, the tensor $L(h_i, \bar{h}_i)$ for geodesically equivalent metrics $h_i \sim \bar{h}_i$ discussed above is $\lambda_i \cdot \text{Id}$. If the multiplicity k_i of λ_i is ≥ 2 , then by the classical result of Weyl [We1] the metrics h_i and \bar{h}_i are proportional, i.e., $\bar{h}_i := \frac{1}{\lambda_i^{k_i+1}} h_i$, where the eigenvalue λ_i is a constant.

If the multiplicity k_i of λ_i is 1, then h_i is one-dimensional and we can obviously choose x_i in such a way that $h_i = dx_i^2$ and $\bar{h}_i = \frac{1}{\lambda_i(x_i)^2} dx_i^2$.

Without loss of generality we can assume that the first r eigenvalues $\lambda_1, \dots, \lambda_r$ have multiplicity 1, and the last $m - r$ eigenvalues have multiplicity ≥ 2 . Then, in the chosen coordinate system, the direct product metrics $h = h_1 + h_2 + \dots + h_m$ and $\bar{h} = \bar{h}_1 + \bar{h}_2 + \dots + \bar{h}_m$ are given by

$$\begin{aligned} ds_h^2 &= \sum_{i=1}^r dx_i^2 + \sum_{i=r+1}^m \left[\sum_{\alpha_i, \beta_i=1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \\ ds_{\bar{h}}^2 &= \sum_{i=1}^r \frac{1}{\lambda_i(x_i)^2} dx_i^2 + \sum_{i=r+1}^m \left[\frac{1}{\lambda_i^{k_i+1}} \sum_{\alpha_i, \beta_i=1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right]. \end{aligned}$$

Here the functions λ_i are constant for $i > r$ and depend only on the corresponding variable x_i for $i \leq r$. The metrics h_i , $i = r + 1, \dots, m$ can be arbitrary, but their entries $(h_i)_{\alpha_i \beta_i}$ must depend on the coordinates $x_i = (x_i^1, \dots, x_i^{k_i})$ only.

Applying Proposition 1 and formula (12), we obtain for g and \bar{g} the following form:

$$\begin{aligned} ds_g^2 &= \sum_{i=1}^r P_i dx_i^2 + \sum_{i=r+1}^m \left[P_i \sum_{\alpha_i, \beta_i=1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \\ ds_{\bar{g}}^2 &= \sum_{i=1}^r P_i \rho_i dx_i^2 + \sum_{i=r+1}^m \left[P_i \rho_i \sum_{\alpha_i, \beta_i=1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right], \end{aligned}$$

where

$$P_i := \pm \prod_{j \neq i} (\lambda_i - \lambda_j), \quad \rho_i := \pm \frac{1}{\lambda_i \prod_{\alpha} \lambda_{\alpha}}. \quad (13)$$

(the signs \pm should be chosen so that all P_i and ρ_i are positive). This is precisely the Levi-Civita normal form from [LC] for geodesically equivalent Riemannian metrics!

As we mentioned above, the results by Levi-Civita and Aminova hold in a neighborhood of almost every point. More precisely, such a point $p \in M$, which we call *regular*, is characterised by the property that the structure of the Jordan normal form of L (the number of Jordan blocks and their dimensions) is the same for all points in some neighborhood $U(p)$. Regular points form an open everywhere dense subset of M . The other points will be called *singular*.

For global questions (in particular, for the description of the topology of closed manifolds admitting geodesically equivalent metrics), it is also necessary to solve Beltrami problem near singular points. For the Riemannian case, it was done in [Ma1, Ma4, Ma5]. The result essentially used the splitting construction: arguing as above, the problem was reduced to a few cases of simple bifurcations, which were considered separately. We expect the same application of the splitting construction in the pseudo-Riemannian case.

2.2 Motivation II. Projective transformations: Lie problem and Lichnerowicz conjecture

For Riemannian manifolds, splitting construction found an important application in the theory of projective transformations. Projective transformations is a very classical object of study. The first examples are due to Beltrami [Be]; as we mentioned at the beginning of Section 2, the problem of local description of projective transformations was explicitly stated by Lie [Lie]. In the global setting, namely under the assumption that (M, g) is complete, the problem was formulated by Schouten [Sc]; the Lichnerowicz conjecture mentioned above is a hypothetical solution of this problem.

In this section we assume that $\dim M \geq 3$. The reason for this is that in dimension 2 the Lie problem was solved in [BMM, Ma6], where a complete list of local metrics admitting projective vector fields was constructed. The list is pretty simple (explicit

formulas involving only elementary functions) and it should not be very complicated to understand which metrics from this list can be prolonged up to a complete metric.

As it was observed by Fubini and Solodovnikov, in dimension ≥ 3 , the degree of mobility plays a crucial role in the description of projective transformations of a given manifold. Recall that *the degree of mobility* of a metric g is the dimension of the space of the solutions of the equation (21) considered as an equation on L . The degree of mobility has a clear geometrical meaning: locally, it coincides with the dimension of the set of metrics geodesically equivalent to g equipped with the natural topology.

In particular, the condition “degree of mobility = 1” means that g does not admit any geodesically equivalent “partner” except for $\bar{g} = \text{const} \cdot g$. In this case projective transformations are just homotheties of g .

Riemannian metrics (on $M^{n \geq 3}$) with degree of mobility greater or equal to three, and their projective vector fields were locally described by Solodovnikov [So]; his result was improved by Shandra [Sh], who obtained a much shorter description based on a certain trick. The results of [KM2] show that

- the metrics with degree of mobility ≥ 3 always admit projective vector fields,
- the trick used in [Sh] survives for pseudo-Riemannian metrics as well, and gives a description of all metrics with degree of mobility ≥ 3 , and their projective vector fields.

Thus, in order to get the complete solution of the Lie Problem in dimension ≥ 3 , it is sufficient to consider metrics with degree of mobility 2. It turns out that under this assumption, the problem can be reduced to the analysis of a certain 1st order system of PDEs. This system is universal in the sense that it does not depend on the metric and remains the same if we pass from the whole manifold M to the components of smaller dimensions obtained by means of the splitting construction. We describe this idea in brief below.

First of all we notice that in all our considerations a pair $g \sim \bar{g}$ of geodesically equivalent metrics can be replaced by the pair (g, L) where L is the (1,1)-tensor defined by (1). Indeed, \bar{g} can be uniquely reconstructed from g and L as $\bar{g} = \frac{1}{\det(L)}gL^{-1}$. We shall say that L and g are *compatible*, if g and $\bar{g} = \frac{1}{\det(L)}gL^{-1}$ are geodesically equivalent.

The $(1, 1)$ -tensors L compatible with g form a finite-dimensional vector space whose dimension is exactly the degree of mobility of g .

Now let v be a projective vector field for the metric g (we work in a small connected neighborhood U of a point $p \in M$). Consider the tensor

$$\tilde{L} := g^{-1}\mathcal{L}_v g - \frac{1}{n+1}\text{tr}(g^{-1}\mathcal{L}_v g) \cdot \text{Id} \quad (14)$$

where \mathcal{L}_v denotes the Lie derivative with respect to v .

Lemma 1 ([Ma4]). *The tensor \tilde{L} given by (14) is compatible with g .*

Lemma 1 is an easy corollary of the compatibility condition (21) below, its proof can be found for example in [Ma4, Section 2.1], see Theorem 3 there. In an equivalent form the statement appeared already in Fubini [Fu] (for dimension 3) and Solodovnikov [So] (under additional assumptions).

Now assume that the degree of mobility of g is 2, so that the vector space of admissible $(1, 1)$ -tensors is two-dimensional. Clearly, the identity tensor Id is admissible and we can take it as the first basis vector. We choose the second one, denoted by L , in such a way that $\det L \neq 0$ in some neighborhood of $p \in M$.

Lemma 2. *Under the above assumptions, the tensor L and the projective vector field v satisfy the following equation*

$$\mathcal{L}_v L = \alpha \cdot L^2 + \beta \cdot L + \gamma \cdot \text{Id}, \quad (15)$$

where β is a function given by $\beta := -\mathcal{L}_v(\log |\det L|) \cdot L + \alpha \cdot \text{tr}(L) + \gamma \cdot \text{tr}(L^{-1}) + \nu$, and α, β, ν are some constants.

Proof. Consider the metric $\bar{g} = \frac{1}{\det(L)}gL^{-1}$. This metric is geodesically equivalent to g and therefore v is a projective vector field also for \bar{g} . Clearly, the degree of mobility of \bar{g} coincides with that of g . Hence, the space of \bar{g} -compatible $(1, 1)$ -tensors is two-dimensional and is generated by Id and L^{-1} .

Thus, applying Lemma 1 to g and \bar{g} , we have:

$$\begin{aligned} g^{-1}\mathcal{L}_v g - \frac{1}{n+1}\text{tr}(g^{-1}\mathcal{L}_v g) &= a \cdot L + b \cdot \text{Id}, \\ \bar{g}^{-1}\mathcal{L}_v \bar{g} - \frac{1}{n+1}\text{tr}(\bar{g}^{-1}\mathcal{L}_v \bar{g}) &= c \cdot L^{-1} + d \cdot \text{Id}, \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ are some constants. It is easy to see that these equations can be rewritten in the following equivalent form:

$$g^{-1}\mathcal{L}_v g = a \cdot L + (a \cdot \operatorname{tr}(L) + (n+1)b) \cdot \operatorname{Id}, \quad (16)$$

$$\bar{g}^{-1}\mathcal{L}_v \bar{g} = c \cdot L^{-1} + (c \cdot \operatorname{tr}(L^{-1}) + (n+1)d) \cdot \operatorname{Id}, \quad (17)$$

On the other hand, we have

$$\begin{aligned} \bar{g}^{-1}\mathcal{L}_v \bar{g} &= \det L \cdot L g^{-1} \mathcal{L}_v \left(\frac{1}{\det L} g L^{-1} \right) = \\ &= -\mathcal{L}_v(\log |\det L|) \cdot \operatorname{Id} + L g^{-1} \mathcal{L}_v g L^{-1} - \mathcal{L}_v(L) \cdot L^{-1} \end{aligned}$$

Substituting (17) to the left hand side and (16) to the right hand side (into the middle term) of this relation, we get

$$\begin{aligned} -\mathcal{L}_v(\log(|\det(L)|)) \cdot \operatorname{Id} + a \cdot L + (a \cdot \operatorname{tr}(L) + (n+1)b) \cdot \operatorname{Id} - \mathcal{L}_v(L) \cdot L^{-1} &= \\ &= cL^{-1} + (c \cdot \operatorname{tr}(L^{-1}) + (n+1)d) \cdot \operatorname{Id}. \end{aligned}$$

After multiplication by L , this relation can be rewritten as

$$\mathcal{L}_v(L) = a \cdot L^2 + (-\mathcal{L}_v(\log |\det L|) + a \operatorname{tr} L - c \operatorname{tr} L^{-1} + (n+1)(b-d)) \cdot L - c \cdot \operatorname{Id},$$

which is equivalent to (15). □

Lemma 3. *Let $\chi = \chi_1 \cdot \chi_2$ be an admissible factorisation of the characteristic polynomial of L and $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ be the local coordinate system induced by this factorisation (see Theorems 1, 2). Then the projective vector field v splits in this coordinate system in the sense that*

$$v = \sum_{i=1}^r v^i(x) \frac{\partial}{\partial x^i} + \sum_{j=r+1}^n v^j(y) \frac{\partial}{\partial y^j},$$

i.e., the first r entries of v do not depend on the y -coordinates, and the last $n-r$ entries of v do not depend on the x -coordinates. Moreover, the function β from Lemma 2 is constant.

Proof. By Theorems 1, 2, in the coordinate system (\bar{x}, \bar{y}) the matrix of L has the block-diagonal form (4). Then, the matrices of L^2 and of $\mathcal{L}_v(\log |\det L|) \cdot L$ are block-diagonal as well with the same dimensions of blocks implying that $\mathcal{L}_v(L)$ is also block-diagonal. Then, the first r components of v depend on x -coordinates only, and the last $(n - r)$ -components of v depend on y -coordinates only: $v = (v_1(x), v_2(y))$ where $v_1 \in TB_1$ and $v_2 \in TB_2$. Multiplying the equation (18) by L^{-1} and rewriting it in the form

$$(\mathcal{L}_v L) \cdot L^{-1} - \alpha \cdot L - \gamma \cdot L^{-1} = \beta \cdot \text{Id}$$

we see that the first block of the left-hand side is independent of y and the second block of the left-hand side is independent of x implying $\beta = \text{const}$. \square

Thus, under the additional assumption that an admissible factorisation exists, the tensor L and the vector field v satisfy the following equation

$$\mathcal{L}_v(L) = \alpha \cdot L^2 + \beta \cdot L + \gamma \cdot \text{Id}, \quad (18)$$

where α , γ and β are certain constants. Moreover, in the notation of Theorem 2, the $r \times r$ - resp. $(n - r) \times (n - r)$ - matrices L_1 and L_2 viewed as $(1, 1)$ -tensors satisfy the equations

$$\mathcal{L}_{v_1}(L_1) = \alpha \cdot L_1^2 + \beta \cdot L_1 + \gamma \cdot \text{Id}, \quad (19)$$

$$\mathcal{L}_{v_2}(L_2) = \alpha \cdot L_2^2 + \beta \cdot L_2 + \gamma \cdot \text{Id}, \quad (20)$$

where the components of the vector v_1 are the first r entries of v , and components of the vector v_2 are the last $n - r$ entries of v . In other words, equation (18) splits into two independent equations (19,20).

We can go further: if χ is a product of several mutually prime monic polynomials χ_1, \dots, χ_m , we can split the equation (20) into m equations of the similar form by repeating the above construction. After finitely many steps, we land at an independent system of PDE of the form

$$\mathcal{L}_{v_i}(L_i) = \alpha \cdot L_i^2 + \beta \cdot L_i + \gamma \cdot \text{Id}, \quad i = 1, \dots, m$$

where each L_i and v_i depend on the corresponding coordinates only. Moreover, in a neighborhood of a regular point we may assume that L_i has one real eigenvalue, or two complex-conjugate eigenvalues.

Thus, it is sufficient to solve equation (18) under the assumption that L has one real eigenvalue, or two complex-conjugate eigenvalues.

In the Riemannian case, equation (18) was obtained by other methods and played an important role in the proof of the projective Lichnerowicz conjecture [Ma4], and in the local description of projective vector fields [Fu, So, BMM, Ma6]. This equation is relatively simple and can be solve in the Riemannian case explicitly. Analysis of its solutions was, in fact, one of the most principal steps in the solution of the Lie and Schouten problems (under the additional assumption that the degree of mobility is at most two).

We expect similar applications in the pseudo-Riemannian case. If L is semi-simple and has real eigenvalues, then at least locally there is no essential difference from the Riemannian case. The possible difficulties might appear if L has Jordan blocks or complex-conjugated eigenvalues. They appear already in dimension 2 (though in dimension 2 the difficulties have been overcome): two dimensional metrics admitting projective vector fields in the case when L is a Jordan block are given by much more complicated formulas than those which appear for L semi-simple, see [Ma6, Theorem 1].

2.3 Motivation III. Topological geodesic rigidity problem

A local version of the splitting/gluing constructions is sufficient for local problems such as finding normal forms for a pair of geodesically equivalent metrics and the Lie problem. For global problems (when the underlying manifold is assumed to be closed or/and complete) such as the Lichnerowicz conjecture discussed above, a general version of the splitting/gluing constructions was useful in the Riemannian case, and is expected to be useful in the pseudo-Riemannian one. Another example of global problems is

Topological geodesic rigidity problem. *On what manifolds, the unparametrized geodesics of any metric g determine this metric uniquely (up to multiplication by a constant)?*

In other words, we want to know whether or not a given manifold M admits at least one pair $g \sim \bar{g}$ of non-proportional geodesically equivalent (pseudo)-Riemannian metrics.

All results in this area are, in fact, very recent: the first obstructions that prevent a

manifold to possess such metrics were found in [MT] (it was proved that geodesically equivalent metrics on a closed surface of genus ≥ 2 are always proportional). Moreover, it was generally believed and explicitly stated in the survey [Mi2] that it is hard to obtain such obstructions. Main research was concentrated in the following direction: find out what metrics are *geodesically rigid*, in the sense they do not possess nontrivial geodesically equivalent “partners”. One of the first results in this direction is [Si2]: every metric geodesically equivalent to an irreducible symmetric metric of nonconstant curvature is proportional to it. Local geodesic rigidity problem was very popular in 60th–80th; there are more than 100 papers devoted to it, see surveys [Mi2, Am2].

Later, one started to investigate the global geodesic rigidity problem (assuming that M is closed or g is complete). A typical result is as follows: if g is a Riemannian Einstein irreducible metric of nonconstant curvature on a closed manifold, then it is geodesically rigid [Mi2, Mi1] (though locally there exist Einstein Riemannian metrics of nonconstant curvature that are not geodesically rigid). The most standard (de-facto, the only) way to prove such results was to use tensor calculus to canonically obtain a nonconstant function f such that $\Delta_g f = \text{const} \cdot f$ with $\text{const} \geq 0$, which of course cannot exist on a closed Riemannian manifold.

Here is one of examples that shows how the splitting/gluing construction works in global setting without any specific assumptions about the metric (we simply combine Theorem 3 with the results by Wu [Wu] and De Rham [DR]).

Proposition 2. *Let M admit a pair of geodesically equivalent (pseudo)-Riemannian metrics g and \bar{g} , one of which is complete. Assume that the characteristic polynomial χ of $L = L(g, \bar{g})$ possesses an admissible global factorisation $\chi = \chi_1 \cdot \chi_2$ on M . Then M admits a (pseudo)-Riemannian metric h with a reducible holonomy group and its universal cover (\widetilde{M}, π^*h) (where $\pi : \widetilde{M} \rightarrow M$ is the natural projection) is the direct product of two (pseudo)-Riemannian manifolds (M_1, h_1) and (M_2, h_2) such that the fundamental group $\pi_1(M)$ acts on $M_1 \times M_2$ by fiberwise isometries.*

Obviously, the above property is a very strong topological restriction on M . For example, in dimension 2, there are only five connected manifolds of this kind: \mathbb{R}^2 , torus, Klein bottle, Möbius strip and cylinder $\mathbb{R}^1 \times S^1$.

The additional assumption in Proposition 2 about the existence of a global admissible factorisation is, of course, very essential. There are many examples of geodesically

equivalent metrics where this condition fails. It appears, however, that in the Riemannian case the nonexistence of such a factorization implies that the fundamental group of M is finite (assuming that M is closed and admits two non-proportional geodesically equivalent metrics) [Ma1].

Combining these two observations, it was possible to describe all geodesically rigid 3-manifolds [Ma2], and to prove that topologically-hyperbolic manifolds are geodesically rigid, in the sense that two geodesically equivalent metrics on such manifolds are proportional [Ma1].

At the present point, it is not clear whether the second observation could be generalized to the pseudo-Riemannian case, i.e., whether the nonexistence of an admissible factorization implies that the fundamental group is finite. However, the known examples allow us to suggest that compared to the classical Riemannian situation, the topological obstructions to the existence of geodesically equivalent metrics are perhaps even stronger in the pseudo-Riemannian case. We conclude this section by the following

Conjecture. *Let $g \sim \bar{g}$ be geodesically equivalent non-proportional metrics on a closed 3-dimensional manifold M^3 . If at least one of the metrics is complete, and at least one of the metrics has signature $(-, +, +)$, then M is a Seifert manifold with zero Euler class.*

3 Proofs

3.1 Vanishing of Nijenhuis torsion and proof of Theorems 1, 2

Throughout the paper we shall use the following analytic condition for two metrics g and \bar{g} to be geodesically equivalent.

Proposition 3. *Let g, \bar{g} be Riemannian or pseudo-Riemannian metrics on the same manifold M^n . Let $L = L(g, \bar{g})$ be given by (1).*

Then, g and \bar{g} are geodesically equivalent, if and only if

$$\nabla_u L = \frac{1}{2}(u \otimes l + (u \otimes l)^*) \quad \text{for any tangent vector } u, \quad (21)$$

or, in coordinates,

$$\nabla_r L_q^p = \frac{1}{2}(\delta_r^p l_q + g^{ps} l_s g_{rq})$$

where ∇ is the Levi-Civita connection associated with g , $l = d \operatorname{tr} L$ and C^* denotes the operator g -adjoint to C , i.e., $g(Cu, v) = g(u, C^*v)$.

The above proposition and equation (21) are due to Sinjukov [Si1], the self-contained proof can also be found in [BM, EM].

Since \bar{g} can be uniquely reconstructed from g and L as $\bar{g} = \frac{1}{\det L} g L^{-1}$, we may replace the pair of metrics (g, \bar{g}) by the pair (g, L) . For convenience, we shall say that a metric g and a g -self-adjoint nondegenerate $(1, 1)$ -tensor L are *compatible* if they satisfy (21), so that the compatibility of g and L is just rephrasing the fact that g and \bar{g} are geodesically equivalent.

First of all we recall that the compatibility of g and L implies that the Nijenhuis torsion of L vanishes identically, see [BM]. To make our paper self-contained, we recall some basic facts about the Nijenhuis torsion and prove that $N_L \equiv 0$ for any $(1, 1)$ -tensor L satisfying (21).

The *Nijenhuis torsion* of L is the $(1, 2)$ -tensor field defined by

$$N_L(u, v) = L^2[u, v] - L[Lu, v] - L[u, Lv] + [Lu, Lv]$$

where u and v are vector fields on M [Haa]. This definition immediately implies

Lemma 4. *The condition $N_L = 0$ admits the following equivalent forms:*

1. $\mathcal{L}_{Lu}L - L\mathcal{L}_uL = 0$ for any vector field u , where \mathcal{L}_u is the Lie derivative along u ;
2. $(\nabla_{Lu}L - L\nabla_uL)v$ is symmetric with respect to u and v for any vector fields u and v , where ∇ is the Levi-Civita connection of an arbitrary metric g (or more generally, any symmetric connection).

Lemma 5 ([BM]). *If L satisfies (21), then $N_L = 0$.*

Proof. By Lemma 4, we just need to verify that $(\nabla_{Lu}L - L\nabla_uL)v$ is symmetric with respect to u and v . We simply use the compatibility condition (21):

$$\begin{aligned} (\nabla_{Lu}L - L\nabla_uL)v &= \frac{1}{2}((Lu \otimes l)v + (Lu \otimes l)^*v - L(u \otimes l)v - L(u \otimes l)^*v) = \\ &= \frac{1}{2}(l(v) \cdot Lu + g(Lu, v) \cdot g^{-1}(l) - l(v) \cdot Lu - g(u, v) \cdot L(g^{-1}(l))) = \\ &= \frac{1}{2}(g(Lu, v) \cdot g^{-1}(l) - g(u, v) \cdot L(g^{-1}(l))) \end{aligned}$$

Here g^{-1} is viewed as the identification map between T_x^*M and T_xM , in particular, $g^{-1}(l) = \text{grad tr } L$. The symmetry of the last expression with respect to u and v is now evident. \square

The next two statements are well known in folklore, however, we could not find any reference with a short proof.

Lemma 6. *If $N_L = 0$, then $N_{p(L)} = 0$ for any polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$ (with constant real coefficients) and, therefore, $N_{f(L)} = 0$ for any function $f : K \rightarrow \mathbb{C}$ satisfying the assumptions (i-v) of Section 1.3.*

Proof. The latter statement about $f(L)$ follows immediately from the definition of $f(L)$. The proof for a polynomial $p(L)$ is as follows. We use the condition $N_L = 0$ in the form $\mathcal{L}_{Lu}L = L\mathcal{L}_uL$ (see Lemma 4). This identity implies

$$\mathcal{L}_{L^n u}L = \mathcal{L}_{L(L^{n-1}u)}L = L\mathcal{L}_{L^{n-1}u}L = L\mathcal{L}_{L(L^{n-2}u)}L = L^2\mathcal{L}_{L^{n-2}u}L = \dots = L^n\mathcal{L}_uL,$$

and, therefore, by linearity

$$\mathcal{L}_{p(L)u}L = p(L)\mathcal{L}_uL.$$

Thus, we have

$$(\mathcal{L}_{p(L)u} - p(L)\mathcal{L}_u)L = 0$$

Now consider the expression $\mathcal{D} = \mathcal{L}_{p(L)u} - p(L)\mathcal{L}_u$ as a “first order differential operator” which satisfies the obvious property $\mathcal{D}(L^n) = \mathcal{D}(L^{n-1})L + L^{n-1}\mathcal{D}(L)$. Hence, the identity $\mathcal{D}(L) = 0$ immediately implies $\mathcal{D}(p(L)) = 0$, i.e.,

$$(\mathcal{L}_{p(L)u} - p(L)\mathcal{L}_u)p(L) = 0,$$

which is exactly the desired condition $N_{p(L)} = 0$. \square

Lemma 7. *Let $N_L = 0$ and $\chi = \chi_1 \cdot \chi_2$ be an admissible factorisation of the characteristic polynomial of L . Then there exists a local coordinate system $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ such that $\partial_{x^1}, \dots, \partial_{x^r}$ and $\partial_{y^{r+1}}, \dots, \partial_{y^n}$ generate the kernels D_1 and D_2 of the operators $\chi_1(L)$ and $\chi_2(L)$ respectively and*

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}.$$

In particular, the distributions D_1 and D_2 are integrable.

Proof. At each tangent space $T_x M$ we have the natural decomposition $T_x M = D_1 \oplus D_2$ where $D_i = \ker \chi_i(L)$. This decomposition defines two natural projectors P_1 and P_2 onto the subspaces D_1 and D_2 respectively. A simple but important observation is that (locally) these projectors can be viewed as functions $P_1 = f_1(L)$ and $P_2 = f_2(L)$ satisfying the assumptions (i-v) from Section 1.3 (see Example 4).

Thus, by Lemma 6, $N_{P_i} = 0$, and we will use this fact to prove the integrability of D_1 and D_2 . In terms of the projectors P_1 and P_2 , these distributions can obviously be interpreted as $D_1 = \ker P_2$ and $D_2 = \ker P_1$. Let $u, v \in D_1 = \ker P_2$, then the condition $N_{P_2} = 0$ gives

$$P_2^2[u, v] - P_2[P_2 u, v] - P_2[u, P_2 v] + [P_2 u, P_2 v] = P_2^2[u, v] = 0,$$

that is, $[u, v] \in \ker P_2^2 = \ker P_2 = D_1$ which is equivalent to the integrability of D_1 by the Frobenius Theorem. The same is obviously true for D_2 by the same reason.

The integrability of these two distributions is equivalent to the existence of a coordinate system (x, y) such that $\partial_{x^1}, \dots, \partial_{x^r}$ and $\partial_{y^{r+1}}, \dots, \partial_{y^n}$ generates the kernels of the operators $\chi_1(L)$ and $\chi_2(L)$. In particular, the operator L in this coordinate system has a block diagonal form:

$$L(x, y) = \begin{pmatrix} L_1(x, y) & 0 \\ 0 & L_2(x, y) \end{pmatrix}$$

Whithout loss of generality we may assume that $\det L \neq 0$, otherwise we can (locally) replace L by $L + c \cdot \text{Id}$ where c is an appropriate constant.

Notice that the operator

$$LP_1 = \begin{pmatrix} L_1(x, y) & 0 \\ 0 & 0 \end{pmatrix},$$

being a function of L , has zero Nijenhuis torsion. Thus, for $u = \partial_{y_\alpha} \in D_2 = \ker P_1$ we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \mathcal{L}_{LP_1 u}(LP_1) - LP_1 \mathcal{L}_u(LP_1) = LP_1 \mathcal{L}_u(LP_1) = \\ &= \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{y_\alpha} L_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_1 \partial_{y_\alpha} L_1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Since L_1 is non-degenerate, we conclude that $\partial_{y_\alpha} L_1 = 0$, i.e., $L_1 = L_1(x)$. Similarly, $L_2 = L_2(y)$, as needed. \square

Proof of Theorems 1, 2. The statements of these theorems are straightforward from Lemmas 5, 7. \square

3.2 Proof of the generalised Topalov-Sinjukov Theorem 5

Consider two geodesically equivalent metrics g and \bar{g} and assume that L given by (1) and a function $f : K \rightarrow \mathbb{C}$ satisfy the assumptions (i–v) of Section 1.3. Our goal is to prove Theorem 5, i.e., to show that the metric $g_f := gf(L)$ is compatible with L . To simplify our notation below, we shall denote g_f by \tilde{g} .

We need to verify that the main equation (21) for g and L implies the similar relation for $\tilde{g} = gf(L)$ and L :

$$\tilde{\nabla}_u L = \frac{1}{2} \left(u \otimes l + (u \otimes l)^* \right), \quad (22)$$

where C^* denotes the operator \tilde{g} -adjoint of C , and $\tilde{\nabla}$ is the covariant differentiation with respect to the Levi-Civita connection related to \tilde{g} .

First of all, we rewrite the condition that we should verify in a slightly different way by subtracting (21) from (22):

$$(\tilde{\nabla}_u - \nabla_u)L = \frac{1}{2} \left((u \otimes l)^* - (u \otimes l)^* \right). \quad (23)$$

Now notice that the difference of two covariant derivatives in the left hand side is not a differential operator, but a tensor expression of the form

$$(\tilde{\nabla}_u - \nabla_u)L = T_u L - L T_u,$$

where $(T_u)_j^i = T_{jk}^i u^k$ and $T_{jk}^i = \Gamma_{jk}^i - \tilde{\Gamma}_{jk}^i$ is a $(1, 2)$ -tensor that represents the difference of the two connections related to g and \tilde{g} .

Using the obvious fact that $C^{\tilde{*}} = f(L)^{-1}C^*f(L)$ for any operator C , we can rewrite (23) as:

$$T_u L - L T_u = \frac{1}{2} \left((u \otimes l)^* - f(L)^{-1} (u \otimes l)^* f(L) \right)$$

Finally multiplying both sides by $f(L)$ from the left, we see that the statement of the Topalov–Sinjukov Theorem is equivalent to the following algebraic relation

$$[f(L)T_u, L] = \frac{1}{2}[f(L), (u \otimes l)^*], \quad (24)$$

where $[\cdot, \cdot]$ denotes the standard commutator of linear operators.

To verify it, we compute the tensor T_u explicitly, using the following equation:

$$T_{jk}^s \tilde{g}_{si} + T_{ki}^s \tilde{g}_{sj} = (\tilde{\nabla}_k - \nabla_k) \tilde{g}_{ij} = -\nabla_k \tilde{g}_{ij}$$

This is a system of linear equations w.r.t. $T_{jk}^s \tilde{g}_{si}$ with the well-known unique solution:

$$T_{jk}^s \tilde{g}_{si} = \frac{1}{2} (\nabla_j \tilde{g}_{sk} + \nabla_k \tilde{g}_{sj} - \nabla_s \tilde{g}_{jk}),$$

or, in invariant terms,

$$\tilde{g}(T_u v, w) = \frac{1}{2} \left((\nabla_u \tilde{g})(v, w) + (\nabla_v \tilde{g})(u, w) - (\nabla_w \tilde{g})(u, v) \right) \quad (25)$$

In our particular case, we have:

$$\begin{aligned} \nabla_u \tilde{g} &= g \nabla_u f(L) = g f'_{t=0}(L + t \nabla_u L) = \\ f'_{t=0} \left(L + t \frac{1}{2} (u \otimes l + (u \otimes l)^*) \right) &= \\ \frac{1}{2} g f'_{t=0}(L + t(u \otimes l)) + \frac{1}{2} g f'_{t=0}(L + t(u \otimes l)^*) &= \frac{1}{2} g (A_u + A_u^*), \end{aligned} \quad (26)$$

where $A_u = f'_{t=0}(L + t(u \otimes l))$ and $A_u^* = f'_{t=0}(L + t(u \otimes l)^*)$. The operators A_u and A_u^* are g -adjoint to each other and satisfy the following important property.

Lemma 8. *In the notation above, $A_u^*v = A_v^*u$.*

Proof. If $f(L) = L$ then $A_u^* = (u \otimes l)^* = g^{-1}(l) \otimes g(u)$ and we have $A_u^*v = g(u, v)g^{-1}(l) = A_v^*u$ (here we consider g and g^{-1} as identification operators between the tangent and cotangent spaces).

If $f(L) = L^2$, then $A_u^* = L(u \otimes l)^* + (u \otimes l)^*L$ and $A_u^*v = g(u, v) \cdot Lg^{-1}(l) + g(u, Lv) \cdot g^{-1}(l) = A_v^*u$ (we use, of course, the fact that L is g -self-adjoint).

More generally, for $f(L) = L^n$, we have

$$A_u^*v = \sum_{m=0}^{n-1} g(L^{n-m}u, v) \cdot L^{m-1}g^{-1}(l) = A_v^*u.$$

Thus, the statement holds for any polynomial $f(L)$ and, therefore, for any smooth function f satisfying (i-v) in Section 1.3.

Using these properties, we get a surprisingly simple result by substituting (26) into (25):

$$\tilde{g}(T_u v, w) = \frac{1}{4}(g((A_u + A_u^*)v, w) + g((A_v + A_v^*)u, w)$$

$$-g((A_w + A_w^*)u, v)) = \frac{1}{2}g(A_u^*v, w)$$

In other words, $f(L)T_u = A_u^* = \frac{1}{2}f'_{t=0}(L + t(u \otimes l)_g^*)$.

We are now ready to complete the proof. We use the following simple matrix relation which obviously holds for any L, B and f :

$$0 = [f(L + tB), L + tB]'_{t=0} = [f'_{t=0}(L + tB), L] + [f(L), B],$$

that is,

$$[f'_{t=0}(L + tB), L] = [B, f(L)].$$

In our case $B = (u \otimes l)_g^*$ and $f'_{t=0}(L + tB) = 2f(L)T_u$ which gives exactly (24), as required. \square

3.3 Proof of Theorem 3

Theorems 1, 2 reduce our consideration to the block-diagonal case in the sense that the metric g and operator L both have block-diagonal form in local coordinates $(x, y) = (x^1, \dots, x^r, y^{r+1}, \dots, y^n)$, more precisely

$$g = \begin{pmatrix} g_1(x, y) & 0 \\ 0 & g_2(x, y) \end{pmatrix}, \quad L = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}. \quad (27)$$

Notice first that in such a situation, the main equation (21) has a rather special form. Namely, it can naturally be divided into three parts each of which has its own meaning and can be treated separately (up to some extent).

Index notation convention. For convenience, throughout the rest of the paper we denote the indices for x^1, \dots, x^r by Latin letters $i, j, k, l, m = 1, \dots, r$, and those for y^{r+1}, \dots, y^n by Greek letters $\alpha, \beta, \gamma, \delta = r + 1, \dots, n$. The indices p, q, r, s, t will serve for both cases, i.e., $p, q, r, s, t = 1, \dots, n$.

Let $u \in D_1$, then the main equation (21) can be written in a block form as

$$\begin{pmatrix} \nabla_u L_j^i & \nabla_u L_\alpha^i \\ \nabla_u L_j^\beta & \nabla_u L_\alpha^\beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \left(u^i \frac{\partial \text{tr} L}{\partial x^j} + g^{il} \frac{\partial \text{tr} L}{\partial x^l} u^m g_{mj} \right) & \frac{1}{2} u^i \frac{\partial \text{tr} L}{\partial y^\alpha} \\ \frac{1}{2} g^{\beta\gamma} \frac{\partial \text{tr} L}{\partial y^\gamma} u^m g_{mj} & 0 \end{pmatrix}$$

We rewrite it for each block separately taking into account the block-diagonal form of g and L and the fact that $\frac{\partial \text{tr} L}{\partial x^l} = \frac{\partial \text{tr} L_1}{\partial x^l}$ and $\frac{\partial \text{tr} L}{\partial y^\alpha} = \frac{\partial \text{tr} L_2}{\partial y^\alpha}$:

$$u^k \frac{\partial (L_1)_j^i}{\partial x^k} + u^k \Gamma_{km}^i (L_1)_j^m - u^k \Gamma_{kj}^m (L_1)_m^i = \frac{1}{2} \left(u^i \frac{\partial \text{tr} L_1}{\partial x^j} + (g_1)^{il} \frac{\partial \text{tr} L_1}{\partial x^l} u^m (g_1)_{mj} \right) \quad (28)$$

$$u^k \Gamma_{k\beta}^j (L_2)_\alpha^\beta - u^k \Gamma_{k\alpha}^m (L_1)_m^j = \frac{1}{2} u^j \frac{\partial \text{tr} L_2}{\partial y^\alpha} \quad (29)$$

$$u^k \Gamma_{\gamma k}^\alpha (L_2)_\beta^\gamma - (L_2)_\gamma^\alpha u^k \Gamma_{\beta k}^\gamma = 0 \quad (30)$$

We omit the equation for $\nabla_u L_j^\beta$ because it is obtained from (29) by using the fact that the left hand side and right hand side on (21) are both g -self-adjoint).

First of all, we notice that the Christoffel symbols Γ_{jk}^i ($i, j, k = 1, \dots, r < n$) of the Levi-Civita connection associated with g coincide with those for the metric g_1 defined on the leaves of the integrable distribution D_1 . Thus, the first equation (28) simply means that $g_1(x, y)$ and $L_1(x)$ are compatible on each leaf of D_1 , i.e., for every fixed y .

To simplify (29) and (30), we notice that

$$\Gamma_{\alpha k}^i = \frac{1}{2} g^{im} \frac{\partial g_{mk}}{\partial y^\alpha} \quad \text{and} \quad \Gamma_{\alpha k}^\beta = \frac{1}{2} g^{\beta\gamma} \frac{\partial g_{\gamma\alpha}}{\partial x^k}$$

Substituting these expressions into (29) and (30) and taking into account that $u \in D_1$ is an arbitrary vector, we obtain

$$g^{jm} \frac{\partial g_{mk}}{\partial y^\beta} (L_2)_\alpha^\beta - (L_1)_m^j g^{mi} \frac{\partial g_{ik}}{\partial y^\alpha} = \delta_k^j \frac{\partial \text{tr } L_2}{\partial y^\alpha} \quad (31)$$

and

$$g^{\alpha\beta} \frac{\partial g_{\beta\gamma}}{\partial x^k} (L_2)_\beta^\gamma - (L_2)_\gamma^\alpha g^{\gamma\beta} \frac{\partial g_{\beta\alpha}}{\partial x^k} = 0 \quad (32)$$

Summarizing this discussion and rewriting (31) and (32) in a shorter “matrix” form, we obtain

Proposition 4. *For g and L of block-diagonal form (27), the compatibility equation (21) for $u \in D_1$ is equivalent to the following three conditions:*

Condition 1: $g_1(x, y)$ and $L_1(x)$ are compatible on each leaf of B_1 (here x are coordinates on a leaf, y is a parameter which determines the leaf);

Condition 2:

$$(g_1^{-1} d_y g_1) L_2 - L_1 (g_1^{-1} d_y g_1) = \text{Id}_{D_1} \otimes d_y \text{tr } L_2; \quad (33)$$

Condition 3:

$$(g_2^{-1} d_x g_2) L_2 - L_2 (g_2^{-1} d_x g_2) = 0. \quad (34)$$

As we shall see now, Theorems 3 and 4 are algebraic corollaries of these equations, generalized Topalov-Sinjukov Theorem 5 and vanishing of N_L .

The first part of Theorem 3 states that the metric

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} g_1 \chi_2(L_1)^{-1} & 0 \\ 0 & g_2 \chi_1(L_2)^{-1} \end{pmatrix}$$

is of local product structure. In other words, we need to check that

$$\frac{\partial(h_1)_{lm}}{\partial y^\alpha} = 0 \quad \text{and} \quad \frac{\partial(h_2)_{\alpha\beta}}{\partial x^m} = 0.$$

We shall verify the first condition only, the proof for the second is similar. We start with straightforward computation.

Lemma 9. *Let $h_1 = g_1 \chi_2(L_1)^{-1}$, then*

$$\frac{\partial(h_1)_{lm}}{\partial y^\alpha} = g_{li} \left(g^{ij} \frac{\partial g_{jk}}{\partial y^\alpha} - (\chi_2(L_1)^{-1})_j^i \frac{\partial \chi_2(L_1)_k^j}{\partial y^\alpha} \right) (\chi_2(L_1)^{-1})_m^k \quad (35)$$

or, in coordinate-free form,

$$d_y h_1 = g_1 (g_1^{-1} d_y g_1 - \chi_2(L_1)^{-1} d_y \chi_2(L_1)) \chi_2(L_1)^{-1}. \quad (36)$$

Proof. Thinking of $h_1 = g_1 \chi_2(L_1)^{-1}$ as a product of two $r \times r$ -matrices, we have

$$\begin{aligned} \frac{\partial}{\partial y^\alpha} h_1 &= \frac{\partial}{\partial y^\alpha} (g_1 \chi_2(L_1)^{-1}) = \frac{\partial g_1}{\partial y^\alpha} \chi_2(L_1)^{-1} + g_1 \frac{\partial \chi_2(L_1)^{-1}}{\partial y^\alpha} = \\ &= \frac{\partial g_1}{\partial y^\alpha} \chi_2(L_1)^{-1} - g_1 \chi_2(L_1)^{-1} \frac{\partial \chi_2(L_1)}{\partial y^\alpha} \chi_2(L_1)^{-1} = \\ &= g_1 \left(g_1^{-1} \frac{\partial g_1}{\partial y^\alpha} - \chi_2(L_1)^{-1} \frac{\partial \chi_2(L_1)}{\partial y^\alpha} \right) \chi_2(L_1)^{-1}, \end{aligned}$$

as needed. □

Thus, we need to prove that the expression in brackets in the left hand side of (35) (or (36)) is actually zero. To do so, we need some properties of the differential of the characteristic polynomial of L in the case when N_L vanishes.

Lemma 10. *Let L be a tensor with zero Nijenhuis torsion, $l = d \operatorname{tr} L$ be the differential of $\operatorname{tr} L$ viewed as a covector and $\chi(t) = \det(t \cdot \operatorname{Id} - L)$ the characteristic polynomial of L viewed as a smooth function on M with t as a formal parameter. Then the differential of $\chi(t)$ satisfies the following relation:*

$$d\chi(t) L - t \cdot d\chi(t) = \chi(t) \cdot l \quad (37)$$

Remark 8. In (37), the right and left hand sides are both covectors, i.e. elements of the cotangent space. The expression $d\chi(t) L$ means that we apply the operator L to the covector $d\chi$ using right multiplication. In coordinates, this means $(d\chi(t) L)_s = \frac{\partial \chi(t)}{\partial x^p} L_s^p$. The multiplication denoted by \cdot simply means multiplying a covector by a scalar function.

Proof of Lemma 10. The differential of the polynomial

$$\chi(t) = \det(t \cdot \operatorname{Id} - L) = a_0 + a_1 t + a_2 t^2 + \dots + t^n,$$

is defined to be

$$d\chi(t) = da_0 + t \cdot da_1 + t^2 \cdot da_2 + \dots + t^{n-1} \cdot da_{n-1}.$$

Instead of differentiating each coefficient separately, we are going to differentiate the whole polynomial (thinking of t as a certain constant). For the operators L with $N_L = 0$, the following property is well known (see, for example, [BM, Lemma1]):

$$(d \log |\det L|) L = d \operatorname{tr} L = l$$

Notice that $N_{t \cdot \operatorname{Id} - L} = 0$ and apply the above identity to $\chi(t) = \det(t \cdot \operatorname{Id} - L)$:

$$(d \log |\chi(t)|) (t \cdot \operatorname{Id} - L) = d \operatorname{tr} (t \cdot \operatorname{Id} - L) = -d \operatorname{tr} L = -l.$$

Thus $\frac{1}{\chi(t)} d\chi(t) (L - t \cdot \operatorname{Id}) = l$, which is equivalent to (37). \square

In fact, we need to compute the differential of the characteristic polynomial in the case when $L = L_2$ and t is replaced by L_1 . More precisely we shall need the expression:

$$(d_y \chi_2(L_1))_{j,\alpha}^i = \frac{\partial}{\partial y^\alpha} (\chi_2(L_1))_j^i$$

This object can be naturally viewed as an element of the space $\text{Hom}(D_1, D_1) \otimes D_2^*$. Since (37) is purely algebraic (in the sense that the nature of t is not important) and $t = L_1$ does not depend on y , we can reformulate (37) for our special case as follows:

$$d_y \chi_2(L_1) L_2 - L_1 d_y \chi_2(L_1) = \chi_2(L_1) \otimes l_2 \quad (38)$$

or, in coordinates,

$$(d_y \chi_2(L_1))_{j,\beta}^i (L_2)_\alpha^\beta - (L_1)_k^i (d_y \chi_2(L_1))_{j,\alpha}^k = (\chi_2(L_1))_j^i (l_2)_\alpha,$$

where $l_2 = d \text{tr } L_2$.

Multiplying the both sides of (38) by $\chi_2(L_1)^{-1}$, and using that L_1 commute with $\chi_2(L_1)$, we obtain

Lemma 11. *If $N_L = 0$, then the expression $\chi_2(L_1)^{-1} d_y \chi_2(L_1)$ satisfies the following relation*

$$(\chi_2(L_1)^{-1} d_y \chi_2(L_1)) L_2 - L_1 (\chi_2(L_1)^{-1} d_y \chi_2(L_1)) = \text{Id}_{D_1} \otimes l_2 \quad (39)$$

or, in coordinates,

$$(\chi_2(L_1)^{-1})_m^j (d_y \chi_2(L_1))_{k,\beta}^m (L_2)_\alpha^\beta - (L_1)_m^j (\chi_2(L_1)^{-1})_l^m (d_y \chi_2(L_1))_{k,\alpha}^l = \delta_k^j (l_2)_\alpha.$$

We are now ready to complete the proof of Theorem 3. Let L and g be compatible. Then Condition 2 from Proposition 4 holds. Comparing this condition (33) with the relation (39), we see that $\chi_2(L_1)^{-1} d_y \chi_2(L_1)$ and $g_1^{-1} d_y g_1$ satisfy the same relation. If we fix index k in (33) and (39), we shall see that these relations can be treated as a linear matrix equation of the form

$$X L_2 - L_1 X = C$$

Where X is an (unknown) matrix of dimension $r \times (n - r)$, and L_1, L_2, C are given matrices of dimensions $r \times r$, $(n - r) \times (n - r)$ and $r \times (n - r)$ respectively.

It is a simple fact in Linear Algebra that if L_1 and L_2 have no common eigenvalues, then the solution to this equation is unique for any C . Hence we conclude that

$$\chi_2(L_1)^{-1} d_y \chi_2(L_1) = g_1^{-1} d_y g_1,$$

which immediately implies $\frac{\partial(h_1)_{ij}}{\partial y^\alpha} = 0$ (see Lemma 9).

Analogously (i.e., just by interchanging the distributions D_1 and D_2), we can check that $\frac{\partial(h_2)_{\alpha\beta}}{\partial x^i} = 0$. This means that in the metric

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

the block h_1 depends on the x -coordinates only, and the block h_2 depends on the y -coordinates only, i.e., (h, B_1, B_2) is a local product structure. Thus, the first statement of Theorem 3 is proved.

The second statement of the theorem 3 says that the restrictions of h and \bar{h} on the same leaf of B_1 , i.e., the metrics $h_1(x) = g_1\chi_2(L_1)^{-1}$ and $\bar{h}_1(x) = \frac{1}{\chi_2(0)}g_1\chi_2(L_1)^{-1}$ are geodesically equivalent. Equivalently, we can reformulate this saying that h_1 and L_1 are compatible on each leaf of the foliation F_1 . To prove this fact we only need to notice that on each fixed leaf $\chi_2(L_1)$ is a polynomial (with constant coefficients) in L_1 and, therefore, $\chi_2(L_1)^{-1}$ is a “good” function of L_1 so that the compatibility of $h_1 = g_1\chi_2(L_1)^{-1}$ and L_1 follows from the Theorem 5 and Condition 1 of Proposition 4.

3.4 Proof of Theorem 4

Now we are going to show that the compatibility of the pairs $h_1(x), L_1(x)$ and $h_2(y), L_2(y)$ imply the compatibility of

$$g = \begin{pmatrix} g_1(x, y) & 0 \\ 0 & g_2(x, y) \end{pmatrix} = \begin{pmatrix} h_1\chi_2(L_1) & 0 \\ 0 & h_2\chi_1(L_2) \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix},$$

which is equivalent to the statement of Theorem 4.

We shall verify the compatibility condition (21) for $u \in D_1$ only (the proof for $u \in D_2$ is absolutely similar). Since g and L are of block-diagonal form (27), we may use Proposition 4 and replace (21) by Conditions 1–3.

Condition 1 (i.e., compatibility of $g_1 = h_1\chi_2(L_1)$ and L_1 on leaves of B_1) immediately follows from the Topalov-Sinjukov theorem and compatibility of h_1 and L_1 .

To verify Condition 2, we notice that Lemma 9 and the condition $\frac{\partial}{\partial y^\alpha} h_1(x) = 0$ imply the relation:

$$g_1^{-1} d_y g_1 - \chi_2(L_1)^{-1} d_y \chi_2(L_1) = 0$$

By Lemma 11, $\chi_2(L_1)^{-1} d_y \chi_2(L_1)$ satisfies (39), therefore so does $g_1^{-1} d_y g_1$, which gives exactly Condition 2.

The last relation (34) (Condition 3) is the matrix equation

$$\left[L_2, g_2^{-1} \frac{\partial g_2}{\partial x^k} \right] = 0, \quad \text{for any } k = 1, \dots, r.$$

By our construction, $g_2 = h_2 \chi_1(L_2)$, where $\chi_1(L_2)$ is a polynomial of the form $\chi_1(L_2) = a_0(x) + a_1(x)L_2 + a_2(x)L_2^2 + a_3(x)L_2^3 + \dots$ and neither h_2 nor L_2 depend on x^k . Hence

$$\begin{aligned} g_2^{-1} \frac{\partial g_2}{\partial x^k} &= \chi_1(L_2)^{-1} h_2^{-1} h_2 \left(\frac{\partial a_0}{\partial x^k} + \frac{\partial a_1}{\partial x^k} L_2 + \frac{\partial a_2}{\partial x^k} L_2^2 + \frac{\partial a_3}{\partial x^k} L_2^3 + \dots \right) = \\ &\chi_1(L_2)^{-1} \left(\frac{\partial a_0}{\partial x^k} + \frac{\partial a_1}{\partial x^k} L_2 + \frac{\partial a_2}{\partial x^k} L_2^2 + \frac{\partial a_3}{\partial x^k} L_2^3 + \dots \right) = f(L_2) \end{aligned}$$

where $f(L_2)$ is a function of L_2 (depending on x as a parameter). Thus, (34) holds since $[L_2, f(L_2)] = 0$ for every function f (satisfying assumptions (i–v) of Section 1.3). This completes the proof of Theorem 4. \square

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