ISSN 0001-4346, Mathematical Notes, 2009, Vol. 86, No. 1, pp. 10–18. © Pleiades Publishing, Ltd., 2009.
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A Formal Frobenius Theorem and Argument Shift

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Received July 23, 2008; in final form, November 29, 2008

Abstract—A formal Frobenius theorem, which is an analog of the classical integrability theorem for smooth distributions, is proved and applied to generalize the argument shift method of A. S. Mishchenko and A. T. Fomenko to finite-dimensional Lie algebras over any field of characteristic zero. A completeness criterion for a commutative set of polynomials constructed by the formal argument shift method is obtained.

DOI: 10.1134/S0001434609070025

Key words: *formal Frobenius theorem, argument shift, finite-dimensional Lie algebra, complete commutative set of polynomials.*

1. INTRODUCTION

In this paper, we prove a formal Frobenius theorem, which is the formal counterpart of the classical theorem on the integrability of smooth distributions. We apply this theorem to construct a commutative set of polynomials in the Poisson algebra $P(\mathfrak{g})$ associated with a finite-dimensional Lie algebra $\mathfrak{g}$ over any field $\mathbb{K}$ of characteristic zero and prove a completeness criterion for this set.

The study of Lie algebras over any field in this context is motivated by the proof of the Mishchenko–Fomenko conjecture [1] that the Poisson algebra of any finite-dimensional real or complex Lie algebra contains a complete commutative set of polynomials. This conjecture was proved by Mishchenko and Fomenko for semisimple Lie algebras by using their method of argument shift [2]. In the general case, it was proved by Sadétov [3], who suggested an algebraic construction reducing the problem to a Lie algebra of smaller dimension over a new field (an extension of the initial one); see also [4]. Thus, constructing a complete commutative set in $P(\mathfrak{g})$ even for a Lie algebra over the familiar field of real or complex numbers involves Lie algebras over new fields.

Let $\mathfrak{g}$ be a Lie algebra over any field $\mathbb{K}$. In this case, unlike in the case of a real, a complex, or an algebraic Lie algebra, there is no group G (Lie or algebraic); this prevents us from using the theory of invariants, which is often applied to study various questions related to Lie algebras. For example, in the argument shift method, the "base" functions for constructing a complete commutative set in $P(\mathfrak{g})$ are invariants of the coadjoint representation, i.e., analytic functions constant on the orbits of the representation $\mathrm{Ad}^*\colon G \to \mathfrak{gl}(\mathfrak{g}^*)$. In the case of an arbitrary field, the coadjoint representation of a group is not defined; however, it turns out that objects playing the role of its invariants still can be defined in a natural way. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then, as is well known, an analytic function $f \in \mathscr{A}(\mathfrak{g}^*)$ is invariant if and only if $\mathrm{ad}^*_{df(x)}\, x = 0$. This definition involves only structural constants of the Lie algebra $\mathfrak{g}$, and hence it makes sense for any field $\mathbb{K}$. In the case of an arbitrary field, we must only agree on what is meant by f. We cannot restrict ourselves to only rational functions from $\mathbb{K}(\mathfrak{g}^*)$, as can be done in the case of algebraic Lie algebras thanks to Rosenlicht's theorem, because $\mathfrak{g}$ may be nonalgebraic, and rational invariants may be insufficient for constructing a complete set. On the other hand, it is well known that, in the real or complex case, global invariants may be undefined, and we must consider local invariants,

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which are essentially convergent series. This suggests the natural idea to consider formal series from the ring $\mathbb{K}[[\mathfrak{g}^*]]$ satisfying certain natural conditions (like $\operatorname{ad}^*_{df(x)} x = 0$) as an invariant (to be more precise, a formal invariant) of the coadjoint representation. Using the formal Frobenius theorem, we can prove that, at every regular point $a \in \mathfrak{g}^*$, a "maximal" set of formal invariants always exists. It is easy to show that the set of the homogeneous parts of formal invariants is a commutative set of polynomials in $P(\mathfrak{g})$. We refer to this method for constructing commutative sets of polynomials as the formal argument shift method. In conclusion, we prove a completeness criterion for a set constructed by this method, which is similar to the criterion in [5], [6].

The paper is organized as follows. In Sec. 2, we prove the formal Frobenius theorem, which is a technical result useful for what follows; this theorem is also of independent interest as the formal counterpart of the classical Frobenius theorem. In Sec. 3, we introduce the notion of a formal invariant for any (not necessarily coadjoint) representation of a Lie algebra and prove the existence of a "maximal" set of such invariants. In Sec. 4, we consider the coadjoint representation $\operatorname{ad}^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ of a Lie algebra. Applying results of the preceding sections, we define the set of polynomials $\mathscr{F}_a \subset P(\mathfrak{g})$ consisting of the homogeneous parts of formal invariants and prove its commutativity. The completeness criterion for $\mathscr{F}_a$ (given in Sec. 6) follows almost automatically from a lemma on the hierarchy generated by a pair of skew-symmetric bilinear forms (proved in Sec. 5).

Throughout the paper, we assume that the field $\mathbb{K}$ has characteristic zero; all topological considerations refer to the Zariski topology.

2. A FORMAL FROBENIUS THEOREM

The classical Frobenius theorem is an integrability criterion for distributions on a manifold and can be stated in terms of vector fields as follows (see, e.g., [7]).

Theorem 1 (the Frobenius theorem). *A smooth distribution $\mathscr{D}$ on a manifold M is integrable if and only if the set of vector fields tangent to $\mathscr{D}$ is closed with respect to the commutator of vector fields.*

To obtain a formal analog of the Frobenius theorem, it suffices to replace smooth geometric objects by their formal counterparts. Below we describe the construction in more detail.

Let $\mathbb{K}^n$ be an affine space over a field $\mathbb{K}$. A formal vector field on $\mathbb{K}^n$ is a vector $v = (v^1(x), \dots, v^n(x))$ whose components are formal power series, that is, $v^i \in \mathbb{K}[[x_1, \dots, x_n]]$. The formal commutator of formal vector fields is defined by the standard formula for a commutator.

Definition 1. A *formal distribution* $\mathscr{D}$ on $\mathbb{K}^n$ is the linear span over $\mathbb{K}[[x_1, \dots, x_n]]$ of a set of formal vector fields:

$$\mathscr{D} = \operatorname{span}\{v_1, \dots, v_k\}.$$

The *rank* of a formal distribution is the rank (over $\mathbb{K}[[x_1, \dots, x_n]]$) of the matrix formed by the components of formal vector fields generating this distribution:

$$\operatorname{rank}\mathscr{D} = \operatorname{rank}\Xi(x), \qquad \Xi(x) = \begin{pmatrix} v_1^1(x) & \dots & v_1^n(x) \\ \vdots & \ddots & \vdots \\ v_k^1(x) & \dots & v_k^n(x) \end{pmatrix}.$$

By definition, the constancy of the rank of a distribution means that the rank of the formal matrix $\Xi(x)$ over the ring of formal series is equal to that of the "numerical" matrix $\Xi(0)$ obtained by setting all variables to zero.

Definition 2. The *formal integral* of a distribution $\mathscr{D} = \operatorname{span}\{v_1, \dots, v_k\}$ is defined as a formal series $F \in \mathbb{K}[[x_1, \dots, x_n]]$ whose derivatives along all formal vector fields determining the distribution $\mathscr{D}$ vanish:

$$v_\alpha(F) := \sum v_\alpha^i \frac{\partial F}{\partial x^i} = 0 \qquad \text{for all} \quad \alpha = 1, \dots, k. \tag{1}$$

Definition 3. A formal distribution $\mathscr{D}$ of constant rank r on $\mathbb{K}^n$ is said to be *formally integrable* if there exist $n-r$ formal integrals $\mathscr{D}$ whose differentials are linearly independent at zero.

Theorem 2 (the formal Frobenius theorem). *A formal distribution $\mathscr{D} = \operatorname{span}\{v_1, \dots, v_k\}$ of constant rank on $\mathbb{K}^n$ is formally integrable if and only if all commutators $[v_i, v_j]$ are linear combinations of $v_1, \dots, v_k$ with coefficients from $\mathbb{K}[[x_1, \dots, x_n]]$ (i.e., the distribution $\mathscr{D}$ is closed with respect to commutator).*

Proof. A formal distribution $\mathscr{D}$ of constant rank r always has a basis; i.e., there exist formal vector fields $u_1, \dots, u_r$ such that any element of $\mathscr{D}$ has a unique representation in the form of a linear combination of $u_1, \dots, u_r$ with coefficients from $\mathbb{K}[[x_1, \dots, x_n]]$. Therefore, it is sufficient to prove the theorem in the case where $k = r$ and the formal fields $v_1, \dots, v_k$ form a basis of the distribution $\mathscr{D}$.

Without loss of generality, we assume that the subspace in $\mathbb{K}^n$ generated by the linearly independent vectors $v_1(0), \dots, v_k(0)$ coincides with the linear span of the first k vectors of the standard basis of $\mathbb{K}^n$. Let us pass to a more convenient basis of the distribution. To this end, consider the $k \times k$ matrix A formed by the first k rows and columns of the matrix

$$\Xi = \begin{pmatrix} v_1^1 & \dots & v_1^k & v_1^{k+1} & \dots & v_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_k^1 & \dots & v_k^k & v_k^{k+1} & \dots & v_k^n \end{pmatrix}.$$

Since $\det A(0) \neq 0$, it follows that the inverse matrix A^{-1} is well defined. Therefore, the passage $\Xi \to \Xi' = A^{-1}\Xi$ is equivalent to a change of basis: the formal vector fields $v_1'(x), \dots, v_k'(x)$ that form the new basis are the rows of the matrix

$$\Xi' = \begin{pmatrix} 1 & \dots & 0 & (v')_1^{k+1} & \dots & (v')_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & (v')_k^{k+1} & \dots & (v')_k^n \end{pmatrix}.$$

Clearly, in the proof of the theorem, we can use the vector fields v_i' instead of the fields v_i.

The formal commutators $[v_i', v_j']$ linearly depend on $v_1', \dots, v_k'$ if and only if the formal vector fields $v_1', \dots, v_k'$ pairwise commute. Indeed, the first k components of each commutator $[v_i', v_j']$ always vanish. Hence $[v_i', v_j']$ is a linear combination of $v_1', \dots, v_k'$ if and only if $[v_i', v_j']$ identically vanishes.

Note also that the "nontrivial" components of the formal vector fields $v_1', \dots, v_k'$ (i.e., those with numbers $\geq k+1$) have no constant terms, because, by assumption, the vectors $v_1'(0), \dots, v_k'(0)$ generate the subspace spanned by the first k vectors of the standard basis in $\mathbb{K}^n$.

For convenience, we change the notation as follows:

(1) hereafter we denote the variables $x_1, \dots, x_k, x_{k+1}, \dots, x_n$ by $x^1, \dots, x^k, y^1, \dots, y^{n-k}$;

(2) we use Latin indices for the x variables and Greek letters for the y variables;

(3) the nontrivial components $(v')_i^{k+1}, \dots, (v')_i^n$ of each formal vector field v_i' are denoted by $u_i^1, \dots, u_i^{n-k}$ (the superscripts are Greek letters).

Using the new notation, we can express the commutativity of formal vector fields v_i' and v_j' is coordinates as

$$(\partial_{x^j} u_i^\alpha + u_j^\gamma \partial_{y^\gamma} u_i^\alpha) - (\partial_{x^i} u_j^\alpha + u_i^\gamma \partial_{y^\gamma} u_j^\alpha) = 0, \qquad \alpha = 1, \dots, n-k. \tag{2}$$

Thus, the objective is to prove that the integrability of the distribution $\mathscr{D} = \operatorname{span}\{v_1', \dots, v_k'\}$ is equivalent to condition (2).

A formal series F is an integral of the distribution $\mathscr{D}$ if and only if

$$\Xi'\,dF = 0.$$

An analysis of this matrix equation at zero shows that, without loss of generality, we can seek formal integrals in the form

$$F = y^\beta + G(x,y), \qquad \text{where} \quad G(0,y)=0.$$

Clearly, the differentials of the $n-k$ formal integrals thus obtained are automatically independent at zero, which means the integrability of the distribution. The formal differential equation for G is written in coordinates as

$$\partial_{x^i} G + u_i^\alpha \partial_{y^\alpha} G + u_i^\beta = 0, \qquad i = 1,\dots,k. \tag{3}$$

In the space of formal series, let $F^{(m)}$ denote the standard projection of a series F on the subspace of polynomials of degree $\le m$. It is easy to show that this standard projection has the following simple properties:

(a) $(F^{(m+r)})^{(m)} = F^{(m)}$ for $r \ge 0$;

(b) $\partial_{x^i} F^{(m)} = (\partial_{x^i} F)^{(m-1)}$;

(c) $(u_i^\alpha F)^{(m)} = (u_i^\alpha F^{(m-1)})^{(m)}$ (because u_i^α begins with linear terms);

(d) $(u_i^\alpha \partial_{x^j} F)^{(m)} = (u_i^\alpha \partial_{x^j} F^{(m)})^{(m)}$ (for the same reason).

By using these properties, it is easy to prove that the projection of system (3) has the form

$$\partial_{x^i} G^{(m)} = -(u_i^\alpha(\partial_{y^\alpha} G^{(m-1)} + \delta_\alpha^\beta))^{(m-1)}. \tag{4}$$

Equation (4) implies that system (3) has very simple structure; namely, successively solving it for $m=1$, $m=2$, $m=3$, and so on, we obtain a system of the form

$$\partial_{x^i} G^{(m)} = P_i(x,y)$$

at each step, where $P_i(x,y)$ is the polynomial found at the preceding step.

Lemma 1. *Suppose that, in the system of equations $\partial_{x^i} f(x,y) = P_i(x,y)$, where $i=1,\dots,k$, the $P_i(x,y)$ are known polynomials and $f(x,y)$ satisfies the initial condition $f(0,y)=0$. Then*

(1) *a solution of this system exists if and only if the consistency conditions, namely, $\partial_{x^j} P_i(x,y) = \partial_{x^i} P_j(x,y)$ hold;*

(2) *if a solution exists, then it is unique.*

Obviously, this assertion is purely algebraic and, therefore, does not depend on the field $\mathbb{K}$. For $\mathbb{K}=\mathbb{R}$, Lemma 1 is well known; hence it is valid for any field of characteristic zero.

Importantly, if $G^{(m)}$ is a solution of system (4), then the initial segment of degree $m-1$ in this polynomial is a solution of system (4) obtained at the preceding step (this follows from Lemma 1).

The consistency conditions for system (4) are equivalent to the existence and uniqueness of $G^{(m)}$ for each m, i.e., to the existence and uniqueness of G as a formal series. Calculating $\partial_{x^j} P_i(x,y)$ for system (4) and taking into account the fact that $G^{(m-1)}$ satisfies system (4) with $m-1$ instead of m, we easily see that the consistency condition holds if and only if the expression

$$\big((\partial_{x^j} u_i^\alpha - u_i^\gamma \partial_{y^\gamma} u_j^\alpha)(\partial_{y^\alpha} G^{(m-1)} + \delta_\alpha^\beta) - u_i^\gamma u_j^\alpha \partial_{y^\gamma}\partial_{y^\alpha} G^{(m-1)}\big)^{(m-2)}$$

is symmetric with respect to i and j. The last term is always symmetric; therefore, the consistency conditions are equivalent to the symmetry of the expression $\partial_{x^j} u_i^\alpha - u_i^\gamma \partial_{y^\gamma} u_j^\alpha$, which is in turn equivalent to identity (2). This completes the proof of the formal Frobenius theorem. □

3. FORMAL INVARIANTS OF REPRESENTATIONS

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{K}$, and let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be its any finite-dimensional representation to a vector space V with $\dim V = n$.

Definition 4. A formal series $F \in \mathbb{K}[[x_1, \dots, x_n]]$ is called a *formal invariant* of the representation ρ at a point $a \in V$ if, for every $\xi \in \mathfrak{g}$, we have the formal identity

$$\langle d_x F, \rho(\xi)(a + x)\rangle = 0. \tag{5}$$

It is easy to show that if $F = f_1 + f_2 + \dots$ is a formal invariant, then its differential at zero $df_1 \in V^*$ is always orthogonal to the subspace $V_a = \{\rho(\xi)a \mid \xi \in \mathfrak{g}\}$; i.e., for any $\xi \in \mathfrak{g}$, we have

$$\langle df_1, \rho(\xi)a\rangle = 0.$$

Recall that an element $a \in V$ is said to be *regular* if its stationary subalgebra

$$\mathrm{St}(a) = \{\xi \in \mathfrak{g} \mid \rho(\xi)a = 0\}$$

is of minimal dimension. The set of regular elements is open and dense in V. The following assertion is a corollary of the formal Frobenius theorem.

Theorem 3. *For any representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and any regular element $a \in V$, there exists a set $\{F^{(1)}, \dots, F^{(s)}\}$ of $s = \dim V - \dim \mathfrak{g} + \dim \mathrm{St}(a)$ formal invariants of the representation ρ at the point a whose differentials at zero are linearly independent.*

Proof. Note that a series F is a formal invariant of the representation ρ if and only if F is a formal integral of the distribution

$$\mathscr{D} = \mathrm{span}\{v_\xi(x) = \rho(\xi)(x + a) \mid \xi \in \mathfrak{g}\}.$$

Since the element a is regular, it follows that the distribution $\mathscr{D}$ has constant rank equal to $\dim \mathfrak{g} - \dim \mathrm{St}(a)$. Therefore, according to the formal Frobenius theorem, it suffices to check the inclusion $[v_\xi(x), v_\eta(x)] \in \mathscr{D}$ for any $\xi, \eta \in \mathfrak{g}$. A direct calculation shows that the map $\xi \mapsto \rho(\xi)(x + a)$ is an antihomomorphism from the Lie algebra $\mathfrak{g}$ to the Lie algebra of formal vector fields on V, i.e.,

$$[v_\xi(x), v_\eta(x)] = -\rho([\xi, \eta])(a + x) = -v_{[\xi,\eta]}(x) \in \mathscr{D}.$$

This completes the proof of the theorem. □

Remark 1. Using the proof of the formal Frobenius theorem, we can successively find the homogeneous parts of formal invariants of representations of any finite order by solving systems of linear equations.

4. DEFINITION AND COMMUTATIVITY OF $\mathscr{F}_a$

The Lie structure of $\mathfrak{g}$ induces a Poisson–Lie bracket on the symmetric algebra $S(\mathfrak{g}) \simeq \mathbb{K}[\mathfrak{g}^*]$, which is defined as

$$\{f, g\} = c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \qquad f, g \in S(\mathfrak{g}).$$

This Poisson–Lie structure transforms the symmetric algebra $S(\mathfrak{g})$ into the Poisson algebra $P(\mathfrak{g}) = (S(\mathfrak{g}), \{\,\cdot\,,\,\cdot\,\})$ associated with $\mathfrak{g}$.

Consider the coadjoint representation $\mathrm{ad}^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$. Suppose that $a \in \mathfrak{g}^*$ is a regular element, i.e., its annihilator

$$\mathrm{Ann}(a) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi a = 0\}$$

is of minimal dimension $s = \mathrm{ind}\, \mathfrak{g}$. Then, by Theorem 3 on formal invariants, there exist s formal series $F^{(1)}, \dots, F^{(s)} \in \mathbb{K}[[\mathfrak{g}^*]]$ such that, for any $\xi \in \mathfrak{g}$, we have

$$\langle d_x F^{(j)}, \mathrm{ad}^*_\xi(a + x)\rangle = 0. \tag{6}$$

Moreover, the differentials $dF^{(1)},\dots,dF^{(s)}$ are linearly independent at zero and form a basis in $\mathrm{Ann}(a)$. Let $F^{(j)}=f_1^{(j)}+f_2^{(j)}+\dots$, where each $f_i^{(j)}\in\mathbb{K}[\mathfrak{g}^*]$ is a homogeneous polynomial of degree i. Then (6) can be rewritten in equivalent polynomial form as

$$\operatorname{span}\{df_1^{(1)},\dots,df_1^{(s)}\}=\mathrm{Ann}(a), \tag{7}$$

$$\mathrm{ad}^*_{df_{i+1}^{(j)}}a+\mathrm{ad}^*_{df_i^{(j)}}x=0 \tag{8}$$

for $i=1,2,\dots$ and $j=1,\dots,s$. Let $\mathscr{F}_a$ denote the subset in the Poisson algebra $P(\mathfrak{g})$ consisting of all these polynomials:

$$\mathscr{F}_a=\{f_i^{(j)}\mid j=1,\dots,s,\ i=1,2,\dots\}\subset P(\mathfrak{g}). \tag{9}$$

Remark 2. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, then any series F satisfying identity (6) is the Taylor expansion at zero of the function $f_a(x)=f(a+x)$, where $f(x)\in\mathscr{I}(\mathfrak{g})$ is a locally analytic invariant of the coadjoint representation. In this case, the set of polynomials $\mathscr{F}_a$ is equivalent, from the point of view of functional dependence, to the family

$$\{f(x+\lambda a)\mid f\in\mathscr{I}(\mathfrak{g}),\ \lambda\in\mathbb{K}\}$$

of shifts of invariants (see [2]). For this reason, we say that the set $\mathscr{F}_a\subset P(\mathfrak{g})$ is obtained by the formal method of argument shift.

The following assertion is a reformulation of a theorem on the commutativity of shifts of invariants of the coadjoint representation, which was proved by Mishchenko and Fomenko in [2].

Theorem 4. *The set $\mathscr{F}_a$ is commutative.*

Proof. The commutativity of the set $\mathscr{F}_a$ follows from a very important general construction known as Lenard's scheme [8]. Suppose that $\{\,\cdot\,,\,\cdot\,\}_1$ and $\{\,\cdot\,,\,\cdot\,\}_2$ are two compatible Poisson brackets, i.e., any linear combination $\alpha\{\,\cdot\,,\,\cdot\,\}_1+\beta\{\,\cdot\,,\,\cdot\,\}_2$ is again a Poisson bracket. A sequence of functions $\{f_i\}_{i\in\mathbb{N}}$ is called a *bi-Hamiltonian hierarchy* if

$$\{f_i,\,\cdot\,\}_1=-\{f_{i+1},\,\cdot\,\}_2.$$

On $S(\mathfrak{g})$, together with the standard Lie–Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$, we can consider the bracket $\{\,\cdot\,,\,\cdot\,\}_a$ obtained by "freezing" the argument, which is defined by

$$\{f,g\}_a=c_{ij}^k a_k\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j},\qquad f,g\in S(\mathfrak{g}).$$

A direct calculation shows that the brackets $\{\,\cdot\,,\,\cdot\,\}$ and $\{\,\cdot\,,\,\cdot\,\}_a$ are compatible. Relation (8) essentially means that the polynomials $\{f_i^{(j)}\}_{i\in\mathbb{N}}$ form a bi-Hamiltonian hierarchy for every $j=1,\dots,s$, and $f_1^{(1)},\dots,f_1^{(s)}$ are Casimir functions for the bracket $\{\,\cdot\,,\,\cdot\,\}_a$, that is,

$$\{f_1^{(j)},\,\cdot\,\}_a=0,$$

$$\{f_{i+1}^{(j)},\,\cdot\,\}_a+\{f_i^{(j)},\,\cdot\,\}=0.$$

Thus, the set $\mathscr{F}_a$ consists of s different bi-Hamiltonian hierarchies. Let $f_i^{(\alpha)},f_j^{(\beta)}\in\mathscr{F}_a$; then

$$\{f_i^{(\alpha)},f_j^{(\beta)}\}=-\{f_{i+1}^{(\alpha)},f_j^{(\beta)}\}_a=\{f_{i+1}^{(\alpha)},f_{j-1}^{(\beta)}\}=-\{f_{i+2}^{(\alpha)},f_{j-1}^{(\beta)}\}_a=\dots=-\{f_{i+j}^{(\alpha)},f_1^{(\beta)}\}_a=0,$$

as required. □

Remark 3. It follows from the proof of the formal Frobenius theorem that the set of polynomials $\mathscr{F}_a$ can be constructed in the following "canonical" way. For any linear polynomial $f_1\in\mathrm{Ann}(a)$, the system

$$\mathrm{ad}^*_{df_{i+1}}a+\mathrm{ad}^*_{df_i}x=0$$

with initial conditions

$$\pi \circ f_i \equiv 0, \qquad i \geq 2,$$

where $\pi\colon \operatorname{Ann}^*(a) \to \mathfrak{g}^*$ is the standard projection, has a unique solution (which is a set of polynomials $f_2, f_3, \ldots \in P(\mathfrak{g})$ with $\deg f_i = i$). In this way, we can construct $\dim \operatorname{Ann}(a)$ different hierarchies that constitute $\mathscr{F}_a$.

To prove a completeness criterion for the set $\mathscr{F}_a$, we need the following lemma from linear algebra.

5. A LEMMA ABOUT PAIRS OF SKEW-SYMMETRIC FORMS

Let V be a finite-dimensional space over a field $\mathbb{K}$ of characteristic zero with $\dim V = n$, and let $\mathscr{S}$ be the two-dimensional linear family of skew-symmetric bilinear forms on V generated by two fixed forms A_1 and A_2: $\mathscr{S} = \operatorname{span}\{A_1, A_2\}$. Consider the subset $\mathscr{S}_0$ of $\mathscr{S}$ consisting of generic forms, i.e., of forms having maximal rank $R_0 = \max_{A\in\mathscr{S}} \operatorname{rank} A$. Let L_0 denote the subspace in V generated by the kernels of generic forms:

$$L_0 = \operatorname{span}\{\operatorname{Ker} A,\ A \in \mathscr{S}_0\}.$$

Suppose that A_1 is a generic form, i.e., $A_1 \in \mathscr{S}_0$, and vectors $e_1^0, \ldots, e_r^0$ form a basis in $\operatorname{Ker} A_1$. Suppose also that the vectors e_i^j satisfy the recursive relations

$$A_1(e_i^0) = 0, \quad A_1(e_i^1) = A_2(e_i^0), \quad A_1(e_i^2) = A_2(e_i^1), \quad A_1(e_i^3) = A_2(e_i^2), \quad \ldots. \tag{10}$$

Such a sequence of vectors $\{e_i^0, e_i^1, \ldots\}$ is called a *hierarchy* generated by the pair of forms A_1 and A_2.

Lemma 2. *Let B be any nontrivial form from the family $\mathscr{S}$, and let $\overline{\mathbb{K}}$ be the algebraic closure of the field $\mathbb{K}$. Then*

(1) *the subspace L_0 is isotropic with respect to B; moreover, L_0 is maximal isotropic if and only if any nontrivial linear combination $\lambda_0 A_0 + \lambda_1 A_1$ with coefficients from $\overline{\mathbb{K}}$ has maximal rank R_0;*

(2) $L_0 = \operatorname{span}\{e_i^j,\ i = 1, \ldots, r,\ j = 0, 1, 2, \ldots\}$.

The proof of the first assertion of Lemma 2 for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which readily carries over to the case of any algebraically closed field, can be found in [6]. Lemma 2 is an easy corollary of the following theorem on the reduction of a pair of skew-symmetric forms to a canonical form.

Let I_k denote the identity matrix of order k, and let $J_{k,\mu}$ be the Jordan block of order k with eigenvalue μ, where $k \in \mathbb{N}$ and $\mu \in \mathbb{K}$.

Theorem 5. *Any pair (A_1, A_2) of skew-symmetric bilinear forms on a finite-dimensional space over an algebraically closed field can be decomposed into the direct sum of pairs of forms, each isomorphic to one of the following pairs*:

(1) $\mathscr{H}_{2k,\mu} = (H_1^{(k,\mu)}, H_2^{(k,\mu)})$, *where*

$$H_1^{(k,\mu)} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}, \qquad H_2^{(k,\mu)} = \begin{pmatrix} 0 & J_{k,\mu} \\ -J_{k,\mu}^t & 0 \end{pmatrix};$$

(2) $\mathscr{H}_{2k,\infty} = (H_1^{(k,\infty)}, H_2^{(k,\infty)})$, *where*

$$H_1^{(k,\infty)} = \begin{pmatrix} 0 & J_{k,0} \\ -J_{k,0}^t & 0 \end{pmatrix}, \qquad H_2^{(k,\infty)} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix};$$

(3) *the Kronecker pair* $\mathscr{K}_{2k-1} = (K_1^{(2k-1)}, K_2^{(2k-1)})$; *this pair of forms is defined on a space of dimension* $(2k-1)$ *and the only nonzero pairings in the basis* $(v_0, \dots, v_{2k-2})$ *are*

$$A_1(v_{2l}, v_{2l+1}) = 1, \quad A_2(v_{2l+1}, v_{2l+2}) = 1, \qquad l = 0, \dots, k-2.$$

This remarkable algebraic fact plays an important role in the theory of bi-Hamiltonian systems (see [9]). Its proof can be found in the paper [10] by Gelfand and Zakharevich; see also [11]. In the context of this theorem, a subspace being maximal isotropic is equivalent to the presence of only Kronecker pairs in the canonical form.

6. A COMPLETENESS CRITERION FOR $\mathscr{F}_a$

Let $\mathscr{F} = \{f_1, f_2, \dots\} \subset P(\mathfrak{g})$ be a commutative set of polynomials. It is well known that the number of algebraically independent polynomials in $\mathscr{F}$ cannot exceed $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$. Indeed, the commutativity of $\mathscr{F}$ means that the subspace $d_x\mathscr{F} = \operatorname{span}\{df(x),\ f \in \mathscr{F}\} \subset \mathfrak{g}$ generated by the differentials of functions from $\mathscr{F}$ at the point x is isotropic with respect to the form $A_x\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by $A_x(\,\cdot\,,\,\cdot\,) = \langle x, [\,\cdot\,,\,\cdot\,]\rangle$, and $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$ coincides with the dimension of a maximal isotropic subspace at a generic point $x \in \mathfrak{g}^*$.

Definition 5. A commutative set $\mathscr{F}$ is said to be *complete* if it contains $(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})/2$ algebraically independent polynomials.

Consider the Lie algebra $\mathfrak{g}^{\overline{\mathbb{K}}} = \mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ obtained from $\mathfrak{g}$ by extending the base field $\mathbb{K}$ to its algebraic closure $\overline{\mathbb{K}}$ and take the set of singular elements in the coalgebra $(\mathfrak{g}^{\overline{\mathbb{K}}})^*$, that is,

$$(\mathfrak{g}^{\overline{\mathbb{K}}})^*_{\mathrm{sing}} = \{x \in (\mathfrak{g}^{\overline{\mathbb{K}}})^* \mid \dim \operatorname{Ann}(x) > \operatorname{ind} \mathfrak{g}\}.$$

Theorem 6. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over a field* $\mathbb{K}$ *of characteristic zero, and let* $a \in \mathfrak{g}^*$ *be a regular element. Then the commutative set* $\mathscr{F}_a$ *constructed by the formal argument shift method is complete if and only if*

$$\operatorname{codim}(\mathfrak{g}^{\overline{\mathbb{K}}})^*_{\mathrm{sing}} \geq 2.$$

Proof. The completeness of the commutative set $\mathscr{F}_a$ means that the subspace $d_x\mathscr{F}_a$ is maximal isotropic with respect to the form A_x for any generic point $x \in \mathfrak{g}^*$ (i.e., on a Zariski open set).

By construction, the commutative set $\mathscr{F}_a$ consists of polynomials $f_i^{(j)}$ (with $j = 1, \dots, \operatorname{ind} \mathfrak{g}$ and $i = 1, 2, \dots$) whose differentials satisfy relations (7) and (8). This system of relations (a bi-Hamiltonian hierarchy) means precisely that, at every point $x \in \mathfrak{g}^*$, the sequence $\{df_i^{(j)}\}_{i \in \mathbb{N}}$ is the hierarchy (10) generated by the pair of forms A_a and A_x, and, moreover, the vectors $df_1^{(j)}$ with $j = 1, \dots, s = \operatorname{ind} \mathfrak{g}$ form a basis in $\operatorname{Ker} A_a = \operatorname{Ann}(a)$.

Thus, to the subspace $d_x\mathscr{F}_a$ Lemma 2 applies, which gives a criterion for this subspace to be maximal isotropic. According to this lemma, $d_x\mathscr{F}_a$ is maximal isotropic if and only if all nontrivial linear combinations of the form $\lambda A_a + \mu A_x = A_{\lambda a + \mu x}$ with coefficients $\lambda, \mu \in \overline{\mathbb{K}}$ have maximal rank $R_0 = \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$. In terms of the Lie algebra $\mathfrak{g}^{\overline{\mathbb{K}}}$, this condition is equivalent to the regularity of all elements of the form $\lambda a + \mu x$ (except the trivial combination). Geometrically, this means that the two-dimensional plane generated by the covectors x and a in $(\mathfrak{g}^{\overline{\mathbb{K}}})^*$ intersects the singular set $(\mathfrak{g}^{\overline{\mathbb{K}}})^*_{\mathrm{sing}}$ only in zero.

The set $\mathscr{F}_a$ is complete if this geometric condition holds for almost all $x \in \mathfrak{g}^*$. Clearly, this happens if and only if $\operatorname{codim}(\mathfrak{g}^{\overline{\mathbb{K}}})^*_{\mathrm{sing}} \geq 2$, which proves the theorem. □

Remark 4. The completeness condition $\operatorname{codim}(\mathfrak{g}^{\overline{\mathbb{K}}})^*_{\text{sing}} \geq 2$ admits a natural interpretation not engaging the algebraic closure of the base field. The point is that the sets of singular elements in $\mathfrak{g}^*$ and in $(\mathfrak{g}^{\overline{\mathbb{K}}})^*$ are determined by the same system of polynomial equations. The corresponding polynomials are minors of order $\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$ of the matrix of the skew-symmetric form A_x, whose elements have the form $(A_x)_{ij} = c^k_{ij} x_k$. Clearly, these minors are polynomials of degree $\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$ in $x_1, \dots, x_n$ with coefficients from $\mathbb{K}$, and the completeness condition means precisely that the greatest common divisor of these polynomials is trivial. Obviously, the greatest common divisor of polynomials does not change under the extension of the field. Note also that the greatest common divisor can be found by Euclid's algorithm.

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