ISSN 1560-3547, Regular and Chaotic Dynamics, 2009, Vol. 14, Nos. 4–5, pp. 431–454. © Pleiades Publishing, Ltd., 2009.

ARTICLES

Bi-Hamiltonian Structures and Singularities of Integrable Systems

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Received May 19, 2009; accepted June 10, 2009

Abstract—A Hamiltonian system on a Poisson manifold M is called integrable if it possesses sufficiently many commuting first integrals $f_1, \dots f_s$ which are functionally independent on M almost everywhere. We study the structure of the singular set K where the differentials $df_1, \dots, df_s$ become linearly dependent and show that in the case of bi-Hamiltonian systems this structure is closely related to the properties of the corresponding pencil of compatible Poisson brackets. The main goal of the paper is to illustrate this relationship and to show that the bi-Hamiltonian approach can be extremely effective in the study of singularities of integrable systems, especially in the case of many degrees of freedom when using other methods leads to serious computational problems. Since in many examples the underlying bi-Hamiltonian structure has a natural algebraic interpretation, the technology developed in this paper allows one to reformulate analytic and topological questions related to the dynamics of a given system into pure algebraic language, which leads to simple and natural answers.

MSC2000 numbers: 37K10, 37J35, 37J20, 17B80

DOI: 10.1134/S1560354709040029

Key words: integrable Hamiltonian systems, compatible Poisson structures, Lagrangian fibrations, bifurcations, semisimple Lie algebras

Since the pioneering work by Franco Magri [1] it has been well known that integrability of many systems in mathematical physics, geometry and mechanics is closely related to their bi-Hamiltonian nature [2–7]. The bi-Hamiltonian structure has been observed in many classical systems and, at the same time, by using the bi-Hamiltonian technics, many new interesting and non-trivial examples of integrable systems have been discovered [8–13]. Moreover, this approach, based on a very simple, natural and elegant notion of compatible Poisson structures, proved to be very powerful in the theory of integrable systems not only for finding new examples, but also for explicit integration and description of analytical properties of solutions.

In our paper we would like to show that the bi-Hamiltonian approach might also be extremely effective in the study of singularities of integrable systems, especially in the case of many degrees of freedom when using other methods often leads to serious computational problems.

Speaking of singularities of integrable Hamiltonian systems, we mean those integral trajectories which lie outside the set of Liouville tori or, in other words, which belong to the singular set that corresponds to those points where the first integrals of a given system become functionally dependent. The analysis of the system on this set and in its neighborhood is undoubtedly very important because the singular set usually contains the most interesting trajectories, in particular, equilibrium points, and its topological structure is closely related to the bifurcations of Liouville tori, monodromy phenomena and other global effects.

The main idea of the present paper can be formulated as follows: the structure of singularities of a bi-Hamiltonian system is determined by that of the corresponding compatible Poisson brackets. Since in many examples the underlying bi-Hamiltonian structure has a natural algebraic

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interpretation, the technology developed in this paper allows one to reformulated rather non-trivial analytic and topological questions related to the dynamics of a given system into pure algebraic language, which often leads to quite simple and natural answers.

In this paper, we decided to focus on ideas, informal explanations and examples. Details of proofs are sometimes omitted. As main illustrating examples we consider the Euler–Manakov top on so(n) and Mischenko–Fomenko systems on real semisimple Lie algebras. Some results about these systems we mention below were obtained earlier by other methods in [14–18] and we just give them a bi-Hamiltonian interpretation. The paper however contains a number of new results obtained by applying consistently the bi-Hamiltonian approach which, as we believe, will work efficiently in many other cases. We also would like to emphasize that almost all of these new results could be obtained in a more straightforward way by using various (sometimes mysterious) computational tricks. Our approach can be considered as their conceptual explanation.

1. BASICS OF TOPOLOGY OF INTEGRABLE HAMILTONIAN SYSTEMS

By an *integrable Hamiltonian system* we mean a Hamiltonian system $\dot{x} = \operatorname{sgrad} H(x)$ on a symplectic manifold (M^{2n}, ω) which admits $n = \frac{1}{2}\dim M$ independent integrals $F_1, \dots, F_n$ commuting with respect to the Poisson bracket defined by the symplectic structure ω.

Here $\operatorname{sgrad} H$ is the *Hamiltonian vector field* on M associated with a smooth *Hamiltonian function* $H : M \to \mathbb{R}$; in coordinates, $(\operatorname{sgrad} H)^i = (\omega^{-1})^{ij}\dfrac{\partial H}{\partial x^j}$. If $\operatorname{sgrad} H$ is complete, it generates a one-parameter group of diffeomorphisms, which is called the *Hamiltonian flow* of the function H.

We say that $F_1, \dots, F_n$ are *independent* if their differentials $dF_1, \dots, dF_n$ are linearly independent on an open everywhere dense subset in M.

To each integrable Hamiltonian system we can assign the following important objects:

- the *momentum mapping* $\Phi : M^{2n} \to \mathbb{R}^n$, where $\Phi(x) = (F_1(x), \dots, F_n(x))$;
- the *Poisson action* of the Abelian group $\mathbb{R}^n$ generated by the Hamiltonian flows of the integrals $F_1, \dots, F_n$;
- the commutative Poisson *algebra of integrals* $\mathcal{F}$ generated by $F_1, \dots, F_n$;
- the *singular Lagrangian fibration* $\mathcal{L}$ on M (sometimes also called *Liouville foliation*) whose fibers are connected components of common levels $\{F_1(x) = a_1, \dots, F_n(x) = a_n\}$, or equivalently, connected components of $\Phi^{-1}(a)$, inverse images of points $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ under the momentum mapping.

The most important object for this paper is the Lagrangian fibration $\mathcal{L}$.

We shall assume that all regular fibers of $\mathcal{L}$ are compact and, therefore, are Lagrangian tori of dimension n usually called *Liouville tori*.

The singular fibers of $\mathcal{L}$ correspond to those points $x \in M$ where the differentials $dF_1(x), \dots, dF_n(x)$ become linearly dependent, or which is the same, where the rank of $d\Phi(x)$ is less than n. Thus, the singularities of $\mathcal{L}$ are essentially the same as those of the momentum mapping Φ.

Consider the *set of critical points* of the momentum mapping:

$$K = \{x \in M \mid \operatorname{rank} d\Phi(x) < n\}.$$

Its image $\Sigma = \Phi(K) \subset \mathbb{R}^n$ is called the *bifurcation diagram* of Φ.

If $a \notin \Sigma$, then its preimage $\Phi^{-1}(a)$ is a disjoint union of Liouville tori. These tori transform smoothly in M under any continuous change of a outside Σ; however, if a passes through Σ, the Liouville tori undergo a bifurcation.

It is clear that the topological properties of the momentum mapping Φ, its singular set K, and bifurcation diagram Σ keep very important information about qualitative behavior of a given dynamical system both in local and in global. Roughly speaking, they help us to understand and to describe the structure of the fibration $\mathcal{L}$, which, in turn, can be viewed as a portrait of the system and contains almost all qualitative information we usually want to know about the system (number and types of equilibrium points, stability of solutions, bifurcations of tori, Hamiltonian monodromy and so on).

Let us recall some basic notions and terminology related to this subject.

We say that $x \in M$ is a *critical point of corank* k (or, equivalently, *of rank* $(n-k)$) if $\operatorname{rank} d\Phi(x) = n-k$. This condition is equivalent to the fact that the orbit $O(x)$ of the $\mathbb{R}^n$-action generated by the integrals passing through x has dimension $n-k$. A singular fiber L of the Lagrangian fibration $\mathcal{L}$ may contain several orbits of different dimension (the standard situation is that this fiber is a stratified manifold whose strata are those orbits). If $n-k = \min\limits_{x\in L} \dim O(x)$, we shall say that L is a *singularity of corank* k.

First of all, as usual in singularity theory, one distinguishes the class of generic (or non-degenerate) singularities.

We recall this definition first for critical points $x \in M$ of corank n. In other words, we assume that the Hamiltonian vector fields of the integrals $F_1, \dots, F_n$ all vanish at x. From the dynamical viewpoint, such points can usually be characterized as isolated equilibria of the system.

Definition 1. *Let* $\operatorname{rank} d\Phi(x) = 0$. *Then the critical point* x *is called non-degenerate if the Hessians* $d^2F_1(x), \dots, d^2F_n(x)$ *are linearly independent and there exists a linear combination* $\lambda_1 d^2F_1(x) + \dots + \lambda_n d^2F_n(x)$ *such that the roots of its "characteristic polynomial"*

$$\chi(t) = \det\left(\sum_{i=1}^{n} \lambda_i\, d^2F_i(x) - t\cdot\omega\right)$$

are all distinct. In a more abstract terminology, the non-degeneracy condition (for a critical point of corank n*) means that the linearizations of the Hamiltonian vector fields* $\operatorname{sgrad} F_1, \dots, \operatorname{sgrad} F_n$ *at the point* x *generate a Cartan subalgebra in the symplectic Lie algebra* $\operatorname{sp}(T_xM, \omega)$.

It is not hard to generalize this definition to the case of arbitrary rank of $d\Phi(x)$ (see, for example, [19, 20]). In this paper we shall discuss the non-degeneracy property only for critical points of corank n and 1 (Sections 8 and 9 respectively). In the case of corank 1 singularities, one can use the following

Definition 2. *Let* $x \in K$ *be a critical point of corank 1, i.e.,* $\operatorname{rank} d\Phi(x) = n-1$. *This point is called non-degenerate if there exists a function* $f \in \mathcal{F}$ *such that* $df(x) = 0$ *and the linearization of the Hamiltonian vector field* $\operatorname{sgrad} f$ *at* x *has at least one non-zero eigenvalue.*

Equivalently, this condition means that the restriction of f *onto the common level of arbitrary* $n-1$ *independent integrals* $f_1, \dots, f_{n-1} \in \mathcal{F}$ *passing through* x *is a Bott function.*

Non-degenerate critical points of the momentum mapping possess a number of remarkable properties. One of them is the existence of a very simple and natural local normal form, see [21].

Eliasson Theorem. *Let* x *be a non-degenerate critical point of rank* l. *Then in a neighborhood of* x, *there exist symplectic coordinates* $p_1, \dots, p_n$, $q_1, \dots, q_n$ *and a diffeomorphic transformation of the integrals*

$$\widetilde{F}_1 = \widetilde{F}_1(F_1, \dots, F_n), \quad \dots, \quad \widetilde{F}_n = \widetilde{F}_n(F_1, \dots, F_n)$$

such that

$$\widetilde{F}_1 = p_1, \quad \dots, \quad \widetilde{F}_l = p_l,$$

and $\widetilde{F}_i$ *for* $i = l+1, \dots, n$ *has one of the following forms*:

$$\begin{aligned}
&1)\ \ \widetilde{F}_i = p_i^2 + q_i^2 && (\textit{elliptic case}),\\
&2)\ \ \widetilde{F}_i = p_i q_i && (\textit{hyperbolic case}),\\
&3)\ \ \begin{aligned}\widetilde{F}_i &= p_i q_{i+1} - p_{i+1} q_i\\ \widetilde{F}_{i+1} &= p_i q_i + p_{i+1} q_{i+1}\end{aligned} && (\textit{focus-focus case}).
\end{aligned}$$

It follows from this theorem that the tangent space T_xM naturally splits into the direct sum of subspaces:

$$T_xM = V_0 \oplus (V_1 \oplus \cdots \oplus V_s).$$

Here V_0 is the $2l$-dimensional "non-singular" subspace that corresponds to the coordinates $p_1, \ldots, p_l$, $q_1, \ldots, q_l$, and $V_1, \ldots, V_s$ are either two- or four-dimensional subspaces. The two-dimensional subspaces can be of two types: hyperbolic and elliptic. The four-dimensional subspaces correspond to the focus-focus type.

From the point of view of dynamics, these subspaces V_i, $i \neq 0$, have a very natural characterization. If $\widetilde{F} = \widetilde{F}(F_1, \ldots F_n)$ is an integral of the system such that $d\widetilde{F}(x) = 0$, then V_i is an invariant subspace for the linearization of the Hamiltonian vector field $\operatorname{sgrad} \widetilde{F}$ at the point x. It is important that V_i is a common invariant subspace for all such integrals $\widetilde{F}$. Moreover, V_i is the tangent space for a symplectic submanifold invariant under all Hamiltonian flows $\operatorname{sgrad} \widetilde{F}$. Elliptic, hyperbolic and focus-focus cases are distinguished by the type of non-zero eigenvalues of the linearized systems on V_i:

1) real $\lambda, -\lambda$ (hyperbolic type),

2) pure imaginary $i\mu, -i\mu$ (elliptic type),

3) complex $\lambda + i\mu, \lambda - i\mu, -\lambda + i\mu, -\lambda - i\mu$ (focus-focus).

Thus, the Eliasson theorem shows that the local structure of non-degenerate singularities (up to a symplectomorphism) can be uniquely characterized by its *type*, i.e., its (co)rank and the number of elliptic, hyperbolic and focus-focus components.

The analysis of concrete examples of various integrable Hamiltonian systems in physics, geometry and classical mechanics shows that almost all critical points of Φ satisfy the non-degeneracy condition.

We shall say that a singular fiber L of a Lagrangian fibration $\mathcal{L}$ is *non-degenerate* if all of its critical points are non-degenerate. It is a remarkable fact due to Nguyen Tien Zung [20, 22] that non-degenerate singularities admit a very simple topological description in terms of almost direct products.

Before starting any global topological analysis for a specific integrable system we have, as a rule, to deal with the following tasks:

1) describe the set of critical points;

2) verify the non-degeneracy condition for observed singularities;

3) find the type of non-degenerate singularities.

The straightforward approach is just to take the Jacobi matrix

$$d\Phi(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_{2n}} \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial F_2}{\partial x_{2n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_{2n}} \end{pmatrix}, \tag{1.1}$$

compute all its $(n \times n)$-minors, and then find those points where all of them vanish. Then for each critical point we need to analyze the Hessians of the integrals in order to verify non-degeneracy and determine the type of the corresponding singularity.

For two degrees of freedom systems this straightforward approach turned out to be quite successful, but in the case of many degrees of freedom we obviously need to use some additional ideas and structures. It turns out that the property of being bi-Hamiltonian affects the structure of singularities of a system and helps to simplify its topological analysis by reformulating many questions in terms of compatible Poisson brackets.

2. BASIC "BI-HAMILTONIAN" NOTIONS

Recall that a skew-symmetric tensor field $\mathcal{A} = (\mathcal{A}^{ij})$ of type $(2,0)$ on a smooth manifold M is called a *Poisson structure* if the operation on $C^\infty(M)$ defined by

$$\{f, g\} = \mathcal{A}^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad \text{for any } f, g, h \in C^\infty(M).$$

In such a situation, the space of smooth functions $C^\infty(M)$ receives the natural structure of an infinite-dimensional Lie algebra, and $\{\ ,\ \}$ is called a *Poisson bracket*.

The *rank of the Poisson structure* (*bracket*) $\mathcal{A}$ *at a point* x is the rank of the skew-symmetric matrix $\mathcal{A}^{ij}(x)$. Speaking of the *rank of* $\mathcal{A}$ *on the manifold* M as a whole, we mean its rank at a generic point, i.e.,

$$\operatorname{rank} \mathcal{A} = \max_{x \in M} \operatorname{rank} \mathcal{A}^{ij}(x).$$

Below we confine ourselves with real-analytic Poisson structures so that generic points always form an open everywhere dense subset in M.

A function $f : M \to \mathbb{R}$ is a *Casimir function* of a Poisson structure $\mathcal{A}$ if $\{f, g\} \equiv 0$ for any smooth function g. We shall denote the space of such functions by $\mathcal{Z}(\mathcal{A})$.

A Casimir function f can be characterized by the following condition: $df(x) \in \operatorname{Ker} \mathcal{A}(x)$ at each point $x \in M$.

If the Poisson structure $\mathcal{A}$ is degenerate, i.e., $\operatorname{rank} \mathcal{A} < \dim M$, then locally in a neighborhood of a generic point, Casimir functions always exist and the number of functionally independent Casimir functions is exactly the *corank of the Poisson structure* $\operatorname{corank} \mathcal{A} = \dim M - \operatorname{rank} \mathcal{A}$, i.e., the differentials of Casimir functions generate the kernel of $\mathcal{A}(x)$ at generic points x.

Definition 3. *Two Poisson structures* $\mathcal{A}$ *and* $\mathcal{B}$ *are compatible if their sum* $\mathcal{A} + \mathcal{B}$ *(or, equivalently, an arbitrary linear combination with constant coefficients) is again a Poisson structure.*

The essential and non-trivial part of the compatibility condition is that $\mathcal{A} + \mathcal{B}$ satisfies the Jacobi identity. Being rewritten in analytical form, this condition means that the so-called Schouten bracket $\{\{\mathcal{A}, \mathcal{B}\}\}$ of the Poisson structures $\mathcal{A}$ and $\mathcal{B}$ identically vanishes, which amounts to a rather non-trivial system of PDEs. A local description of compatible Poisson brackets can be found in (see [23–25]).

Many integrable dynamical systems in mathematical physics and mechanics possess the property of being *bi-Hamiltonian*, i.e., Hamiltonian with respect to two compatible Poisson brackets $\mathcal{A}$ and $\mathcal{B}$ (or with respect to any non-trivial combination $\lambda'\mathcal{A} + \lambda\mathcal{B}$). This property can be understood as an additional symmetry of a given system which leads to the existence of a big algebra of commuting first integrals. These integrals can be constructed by using the so-called Magri–Lenard scheme (see [1, 6]). In this paper, we discuss one of its versions.

Consider a family (pencil) of compatible Poisson brackets $\mathcal{P} = \{\lambda'\mathcal{A} + \lambda\mathcal{B} \mid \lambda', \lambda \in \mathbb{R}\}$ on a manifold M.

Convention. In this theory, one considers linear combinations $\lambda'\mathcal{A} + \lambda\mathcal{B}$ up to proportionality, so that we may assume that $\lambda' = 1$, but λ may have value ∞. Thus we shall use notation $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}$ (assuming that $\lambda \in \overline{\mathbb{R}}$ or $\lambda \in \overline{\mathbb{C}}$) and shall sometimes refer to $\mathcal{B}$ as $\mathcal{A}_\infty$.

Assume that all $\mathcal{A}_\lambda \in \mathcal{P}$ are degenerate, i.e. $\operatorname{rank} \mathcal{A}_\lambda < \dim M$. By definition, we set the *rank of the pencil* $\mathcal{P}$ to be

$$\operatorname{rank} \mathcal{P} = \max_{\lambda \in \mathbb{R}} \operatorname{rank} \mathcal{A}_\lambda.$$

If $\operatorname{rank} \mathcal{A}_\lambda$ is maximal in the family $\mathcal{P}$, i.e., $\operatorname{rank} \mathcal{A}_\lambda = \operatorname{rank} \mathcal{P}$, we shall say that $\mathcal{A}_\lambda$ is *generic*. Similarly for a point $x \in M$, we can introduce $\operatorname{rank} \mathcal{P}(x)$ as $\max\limits_{\lambda \in \mathbb{R}} \operatorname{rank} \mathcal{A}_\lambda(x)$ and speak about generic

Poisson structures at a given point x. Since the rank of $\mathcal{A}_\lambda$ may vary on M, "*being generic at* x" is not the same as "*being generic on* M". (Usually we shall write $\mathcal{A}_\mu$ for a generic Poisson structure and $\mathcal{A}_\lambda$ for an arbitrary Poisson structure in the pencil.)

The next statement gives a recipe for constructing a large family of commuting functions on M.

Proposition 1. *Let $\mathcal{F}_\mathcal{P}$ be the algebra generated (with respect to usual multiplication of functions) by Casimir functions of all generic Poisson structures $\mathcal{A}_\mu \in \mathcal{P}$.*

1) *$\mathcal{F}_\mathcal{P}$ is commutative with respect to every Poisson structure $\mathcal{A}_\lambda \in \mathcal{P}$.*

2) *If $\dot{x} = v(x)$ is a dynamical system which is Hamiltonian with respect to every generic Poisson structure $\mathcal{A}_\mu \in \mathcal{P}$, then each function from $\mathcal{F}_\mathcal{P}$ is its first integral.*

This statement, however, say nothing about the number N of functionally independent integrals in the family $\mathcal{F}_\mathcal{P}$. Recall that the completeness condition for $\mathcal{F}_\mathcal{P}$ that guarantees the Liouville integrability of $\dot{x} = v(x)$ on M is $N = \frac{1}{2}(\dim M + \operatorname{corank} \mathcal{P})$. This means that the subspace $d\mathcal{F}_\mathcal{P}(x) \subset T_xM$ generated by the differentials $df(x)$, $f \in \mathcal{F}_\mathcal{P}$, is maximal isotropic w.r.t. generic $\mathcal{A}_\mu$ almost everywhere on M.

Another question, even more important in the context of our paper, is to describe those points $x \in M$ where the completeness condition fails, i.e., the dimension of $d\mathcal{F}_\mathcal{P}(x) \subset T_xM$ drops.

A simple and efficient tool which allows us to answer both questions is the following Linear Algebra theorem that describes the canonical form for a one-parameter family (pencil) of skew-symmetric forms (see [26, 27]).

Jordan–Kronecker Theorem. *Let A and B be skew-symmetric bilinear forms defined on a finite-dimensional complex vector space V. Then there exists a basis in V in which the pencil $P = \{A + \lambda B\}$ takes a block-diagonal form*

$$A + \lambda B = \begin{pmatrix} A_1(\lambda) & & & \\ & A_2(\lambda) & & \\ & & \ddots & \\ & & & A_k(\lambda) \end{pmatrix}$$

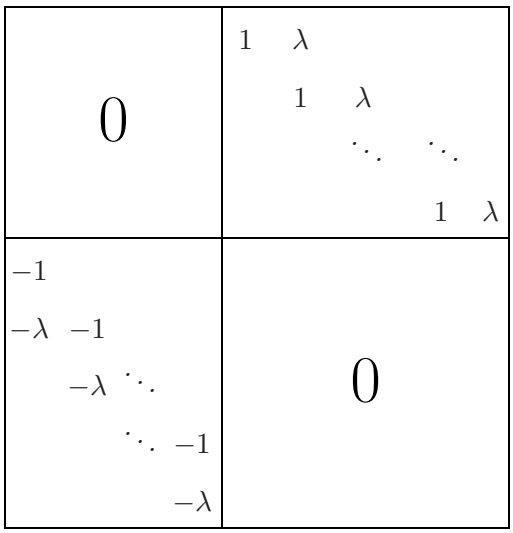
with the blocks $A_i(\lambda)$ of three following types:

$$\left(\begin{array}{c|c} 0 & \begin{matrix} \lambda_i{-}\lambda & 1 & & \\ & \lambda_i{-}\lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i{-}\lambda \end{matrix} \\ \hline \begin{matrix} \lambda{-}\lambda_i & & & \\ -1 & \lambda{-}\lambda_i & & \\ & \ddots & \ddots & \\ & & -1 & \lambda{-}\lambda_i \end{matrix} & 0 \end{array}\right)$$

Jordan block for $\lambda_i \in \mathbb{C}$

$$\left(\begin{array}{c|c} 0 & \begin{matrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & & \\ -\lambda & -1 & & \\ & \ddots & \ddots & \\ & & -\lambda & -1 \end{matrix} & 0 \end{array}\right)$$

Jordan block for $\lambda_i = \infty$



Kronecker block

We also allow trivial (1×1)*-blocks* $A_i(\lambda) = (0)$.

If $\operatorname{rank} A_\mu = \max\limits_{\lambda\in\mathbb{C}} \operatorname{rank} A_\lambda$, then the form A_μ is *generic*. It is clear, that almost all forms in P are generic except for a finite number of singular values of λ for which the rank of A_λ drops. It is easy to see that these singular values of λ, called the *characteristic numbers* of the pencil, are exactly the numbers λ_i (including $\lambda_i = \infty$) that appear in the above theorem. In this theory, characteristic numbers play the same role as eigenvalues in the theory of linear transformations. Moreover, in the "symplectic case" when B is non-degenerate, they are just the eigenvalues of the operator $Q = -AB^{-1}$. Following this analogy, we can naturally introduce the multiplicity of each λ_i as the sum of the sizes of those Jordan blocks where λ_i appears (some of λ_i in the Jordan–Kronecker theorem may, of course, coincide). Notice that the multiplicity is an even number here.

We shall say that the ("Jordan" part of the) pencil $P = \{A + \lambda B\}$ is diagonalizable if the Jordan blocks in the Jordan–Kronecker decomposition are all of minimal size, i.e., 2×2. A sufficient condition for diagonalizability is that the multiplicity of each characteristic number is 2.

Jordan–Kronecker theorem has two simple corollaries important for applications.

Proposition 2. *Let* Z *be the subspace in* V *generated by the kernels of all generic forms* $A_\mu \in P$:

$$Z = \operatorname{span}\{\operatorname{Ker} A_\mu \mid A_\mu \text{ is generic}\}.$$

Then Z *is isotropic with respect to any form* $A_\lambda \in P$.

Notice that Proposition 1 is a straightforward corollary of this purely algebraic fact.

The next statement gives us necessary and sufficient conditions for Z to be *maximal isotropic*.

Proposition 3. *The following properties of the pencil* P *are equivalent*:

1) Z *is a maximal isotropic subspace w.r.t. one* (*distinguished*) *form* $A_{\lambda_0} \in P$;

2) Z *is a maximal isotropic subspace w.r.t. any form* $A_\lambda \in P$, $\lambda \in \overline{\mathbb{C}}$;

3) *the Jordan–Kronecker decomposition contains Kronecker blocks only* (*i.e., no Jordan blocks appear*);

4) $A_\lambda \in P$ *has the same rank for every* $\lambda \in \overline{\mathbb{C}}$ (*in other words,* A_λ*'s are all generic*).

3. STATEMENT OF THE PROBLEM

In this paper, we discuss the following situation. There is a manifold M with a pencil $\mathcal{P} = \{\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}\}$ of compatible Poisson brackets. We assume that $\operatorname{rank}\mathcal{P} < \dim M$ and consider the family $\mathcal{F}_\mathcal{P}$ generated by the Casimir functions of all generic Poisson brackets $\mathcal{A}_\mu \in \mathcal{P}$. This family consists of first integrals of dynamical systems which are bi-Hamiltonian w.r.t. $\mathcal{P}$ (see Proposition 1).

Throughout the paper we suppose that M and $\mathcal{P}$ are real-analytic and satisfy several natural assumptions listed below.

The Casimir functions of every Poisson structure $\mathcal{A}_\lambda$ are globally defined and they distinguish all symplectic leaves of maximal dimension. Sometimes the explicit description of Casimir functions for particular Poisson structures from our pencil may not be so easy. For our purposes it is however sufficient that the above property holds for *almost all* brackets and even for *sufficiently many* brackets. To be more precise, in this paper we shall assume that at least for one generic bracket, say $\mathcal{A} = \mathcal{A}_0$, there exist global Casimir functions $f_1, \dots, f_k$ whose differentials generate the kernel of $\mathcal{A}(x)$ at every point $x \in M$ of maximal rank, i.e.,

$$\operatorname{Ker}\mathcal{A}(x) = \operatorname{span}\{df_1(x), \dots, df_k(x) \mid f_i \in \mathcal{Z}(\mathcal{A})\} \quad \text{if } \operatorname{rank}\mathcal{A}(x) = \operatorname{rank}\mathcal{P}.$$

Moreover, each of these functions $f_i \in \mathcal{Z}(\mathcal{A})$ can be deformed $f_i(x) \mapsto f_{i,\lambda}(x)$ in such a way that $f_{i,\lambda}(x)$ is a globally defined Casimir function for $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}$ at least for small λ with the same property:

$$\operatorname{Ker}\mathcal{A}_\lambda(x) = \operatorname{span}\{df_{1,\lambda}(x), \dots, df_{k,\lambda}(x) \mid f_{i,\lambda} \in \mathcal{Z}(\mathcal{A}_\lambda)\} \quad \text{if } \operatorname{rank}\mathcal{A}_\lambda(x) = \operatorname{rank}\mathcal{P}.$$

The functions $f_{i,\lambda}$ depend on λ smoothly. In particular, we can expand them into Taylor series in λ:

$$f_{i,\lambda}(x) \simeq F_{i,0}(x) + \lambda F_{i,1}(x) + \lambda^2 F_{i,2}(x) + \dots + \lambda^m F_{i,m}(x) + \dots$$

and then take the coefficients $F_{i,m}$ as generators of the commutative family $\mathcal{F}_\mathcal{P}$.

This assumption guarantees that at each point $x \in M$ (such that $\operatorname{rank}\mathcal{P} = \operatorname{rank}\mathcal{P}(x)$) the subspace $d\mathcal{F}_\mathcal{P}(x) = \operatorname{span}\{df(x) \mid f \in \mathcal{F}_\mathcal{P}\}$ generated by the differentials of our first integrals coincides with the subspace in T^*_xM spanned by the kernels of generic $\mathcal{A}_\mu(x)$'s. This property is crucial in all our considerations and allows us to reformulate many phenomena related to bi-Hamiltonian systems in the language of Linear Algebra.

Let us comment on the additional assumption $\operatorname{rank}\mathcal{P} = \operatorname{rank}\mathcal{P}(x)$ that we have just made about a point $x \in M$. All points in M can be divided into three types depending on which of the following conditions holds:

1) $\operatorname{rank}\mathcal{A}_\lambda(x) = \operatorname{rank}\mathcal{P}$ for all $\lambda \in \overline{\mathbb{C}}$;
2) $\operatorname{rank}\mathcal{A}_\lambda(x) = \operatorname{rank}\mathcal{P}$ for almost all $\lambda \in \overline{\mathbb{C}}$ except for a finite number of parameter values $\lambda_1, \dots, \lambda_s$ (recall that these values are called *characteristic numbers* of the pencil $\mathcal{P}$ at the point x);
3) $\operatorname{rank}\mathcal{A}_\lambda(x) < \operatorname{rank}\mathcal{P}$ for all $\lambda \in \overline{\mathbb{C}}$.

As we shall see below, points of the first type are exactly those where $\mathcal{F}_\mathcal{P}$ is complete (i.e., contains sufficiently many independent first integrals). Points of the second type are critical in the sense that the differentials of the functions $f \in \mathcal{F}_\mathcal{P}$ become linearly dependent, but we can control the structure and dimension of the subspace in T^*_xM generated by these differentials. Such points are the main subject of the present paper. Finally, a point $x \in M$ of the third kind is even more singular: the rank of all generic Poisson structures $\mathcal{A}_\lambda$ drops at this point and, therefore, in general we can say nothing about the behavior of Casimir functions in its neighborhood. The analysis of such points is beyond the scope of this paper. By assuming $\operatorname{rank}\mathcal{P} = \operatorname{rank}\mathcal{P}(x)$, we exclude them from our considerations.

The questions we would like to discuss in this paper are:

- *Completeness.* Is $\mathcal{F}_\mathcal{P}$ complete in the sense that the number of independent commuting integrals in $\mathcal{F}_\mathcal{P}$ is sufficient for Liouville integrability?
- *Singular set.* Assume $\mathcal{F}_\mathcal{P}$ is complete. What is the set of those points where the first integrals from $\mathcal{F}_\mathcal{P}$ become dependent?

- *Common equilibria.* What are those points $x \in M$, where all Hamiltonian vector fields $\operatorname{sgrad} f(x)$ vanish simultaneously (i.e., rank 0 singularities in the sense of Section 1)?
- *Non-degeneracy of equilibrium points.* If x is a common equilibrium, how can we check whether or not it is non-degenerate?
- *Corank* 1 *singularities.* These are the most typical singularities which correspond to generic bifurcations of Liouville tori. How can we check the non-degeneracy condition and recognize their type (elliptic or hyperbolic)?

4. TWO BASIC EXAMPLES

In subsequent sections we give answers to all questions formulated at the end of the previous section both in general situation and for Examples A and B described below. So, these examples are "continued" in subsequent sections, and the explanation is organized as follows: each of the next sections contains an answer to the corresponding question in general situation (Theorems 1–5) and "continuations" of Examples A and B with a more concrete form of that answer (Theorems 1A–5A and 1B–5B respectively).

In this section we start our model examples by describing two classical integrable systems, their bi-Hamiltonian structures and integrals.

Example A (Description). The first construction, called *argument shift method*, was suggested by A.T. Fomenko and A.S. Mischenko [17] as a generalization of S.V. Manakov's idea [16], which will be also discussed below (see Example B).

Let $\mathfrak{g}$ be a finite-dimensional (real) Lie algebra and $\mathfrak{g}^*$ its dual space endowed with the standard Lie–Poisson bracket:

$$\{f,g\}(x) = x\big([df(x), dg(x)]\big), \qquad x \in \mathfrak{g}^*, \quad df(x), dg(x) \in (\mathfrak{g}^*)^* = \mathfrak{g}, \tag{4.1}$$

where $f, g : \mathfrak{g}^* \to \mathbb{R}$ are arbitrary smooth functions. Equivalently, in local coordinates this bracket can be written as

$$\{f,g\}(x) = c^i_{jk} x_i \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k},$$

where c^i_{jk} are the structure constants of $\mathfrak{g}$ with respect to the basis associated with the coordinates $x_1, \dots, x_n$.

The Casimir functions of the Lie–Poisson bracket (4.1) are exactly invariants of the coadjoint representation of the corresponding Lie group G, and the coadjoint orbits are symplectic leaves of (4.1). Recall that the codimension of generic orbits (or equivalently, the corank of bracket (4.1)) is called the *index* of $\mathfrak{g}$ and is denoted by $\operatorname{ind}\mathfrak{g}$.

Besides the standard Lie–Poisson bracket, on the dual space $\mathfrak{g}^*$ we can define a constant bracket $\{\ ,\ \}_a$ for any $a \in \mathfrak{g}^*$ by

$$\{f,g\}_a(x) = a\big([df(x), dg(x)]\big) = c^i_{jk} a_i \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}. \tag{4.2}$$

It can be easily verified that (4.1) and (4.2) are compatible. The Casimir functions of a linear combination $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ are of the form $f(x + \lambda a)$, where f is an invariant of the coadjoint representation. Thus we have a family of commuting functions on $\mathfrak{g}^*$:

$$\mathcal{F}_a = \{f(x + \lambda a) \mid f \in \mathrm{I}_{\mathrm{Ad}^*}(G),\ \lambda \in \mathbb{R}\},$$

where $\mathrm{I}_{\mathrm{Ad}^*}(G)$ denotes the ring of coadjoint invariants.

Below we discuss the properties of $\mathcal{F}_a$ for semisimple Lie algebras. In this case $\mathfrak{g}$ and $\mathfrak{g}^*$ can naturally be identified by means of the Killing form so that the adjoint and coadjoint representations will just coincide. It is well known that $\mathrm{I}_{\mathrm{Ad}}(G)$ admits a basis which consists of homogeneous polynomials $f_1, \dots, f_{\operatorname{ind}\mathfrak{g}}$ such that $\sum \deg f_i = \frac{1}{2}(\dim\mathfrak{g} + \operatorname{ind}\mathfrak{g})$. Thus, if we expand each of $f_i(x + \lambda a)$ in powers of λ and take the coefficients as new functions of x, we obtain a

collection of $s = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ commuting homogeneous polynomials (we keep the same notation as above):

$$\mathcal{F}_a = \{F_1, \dots, F_s\}. \tag{4.3}$$

By construction, these functions are first integrals of any dynamical system which is Hamiltonian w.r.t. every bracket $\{\ ,\ \} + \lambda\{\ ,\ \}_a$. The most interesting examples of such systems related to left-invariant geodesic flows on Lie groups were described by A.T. Fomenko and A.S. Mischenko in [17]. One of our goals is to describe the singularities of the momentum mapping

$$\Phi_a : \mathfrak{g} \to \mathbb{R}^s, \qquad \Phi_a(x) = (F_1(x), \dots, F_s(x)), \qquad s = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$$

associated with the brackets (4.1) and (4.2).

Example B (Description). Let $\mathfrak{g} = \mathrm{so}(n)$ be considered as the space of skew-symmetric $(n \times n)$-matrices. As usual, we identify $\mathrm{so}(n)$ and $\mathrm{so}(n)^*$ by means of the Killing form. Along with the standard commutator $[X, Y] = XY - YX$ we introduce on $\mathrm{so}(n)$ another operation:

$$[X, Y]_C = XCY - YCX,$$

where C is a symmetric matrix.

It is easy to see that $[X, Y]_C$ satisfies the Jacobi identity and is compatible with the standard commutator in the sense that any of linear combinations $\lambda[\ ,\ ] + \lambda'[\ ,\ ]_C = [\ ,\ ]_{\lambda E + \lambda' C}$ still defines a Lie algebra structure on $\mathrm{so}(n)$ (considered as the space of skew-symmetric matrices).

Interpreting this observation into the dual language, we may say that on $\mathrm{so}(n) = \mathrm{so}(n)^*$ there is a pencil of compatible Poisson brackets $\{\ ,\ \}_{C+\lambda E}$ related to the commutators $[\ ,\ ]_{C+\lambda E}$.

It is an interesting fact that one of the most famous Hamiltonian systems on $\mathrm{so}(n)$, which describes the n-dimensional rigid body dynamics,

$$\dot{X} = [\phi(X), X], \tag{4.4}$$

where $\phi(X)$ is defined by the relation $[\phi(X), C] = [X, B]$ for some symmetric matrices B and C (where C is assumed to be regular), is Hamiltonian w.r.t. any bracket from the pencil $\{\ ,\ \}_{C+\lambda E}$ (see [3, 28]) and, therefore, possesses the Casimir functions of $\{\ ,\ \}_{C+\lambda E}$ as its first integrals.

It can be easily seen [28, 29] that these Casimir functions can be taken of the form

$$\operatorname{Tr}\bigl(X(C + \lambda E)^{-1}\bigr)^k,$$

and this family of commuting functions in the sense of functional dependence is equivalent to the standard integrals

$$\operatorname{Tr}(X + \lambda C)^k$$

found by S.V. Manakov [16]. Similarly to the argument shift method, this family admits a natural basis that consists of homogeneous polynomials obtained by expanding $\operatorname{Tr}(X + \lambda C)^k$ in powers of λ, where $k = 2, \dots, n$. It is not hard to see that the coefficients of odd degrees in X vanish and this procedure gives us exactly $s = \frac{1}{2}(\dim \mathrm{so}(n) + \operatorname{ind} \mathrm{so}(n))$ non-trivial commuting polynomials. (Remark: Among these functions there is $\operatorname{Tr} X^n$. If n is even, $\operatorname{Tr} X^n$ should be replaced by the Pfaffian of X, that is, $\sqrt{\det X}$). We denote the collection of homogeneous polynomials so obtained by

$$\mathcal{F}_C = \{G_1, \dots, G_s\}. \tag{4.5}$$

Let us emphasize once again that $\mathcal{F}_C$ can, in fact, be treated as the collection of the Casimir functions associated to the pencil $\{\ ,\ \}_{C+\lambda E}$, in particular, each F_i is a first integral of (4.4).

For the integrable system (4.4), our goal is the same as before, i.e., the analysis of singularities of the momentum mapping:

$$\Phi_C : \mathrm{so}(n) \to \mathbb{R}^s, \qquad \Phi_C(X) = (G_1(X), \dots, G_s(X)), \qquad s = \frac{1}{2}(\dim \mathrm{so}(n) + \operatorname{ind} \mathrm{so}(n)). \tag{4.6}$$

Notice that $\mathcal{F}_C$ looks quite similar to the family of shifts of Ad^*-invariants $\mathcal{F}_a$. The only difference is that the matrix C in the case of n-dimensional rigid body lies outside the (co)algebra $\mathrm{so}(n)$, whereas in the classical argument shift method $a \in \mathfrak{g}^*$. However, as we shall see below, from the topological point of view these two classes of dynamical systems are not similar at all. The reason is hidden in the different nature of the two pencils $\{\ ,\ \}_{C+\lambda E}$ and $\{\ ,\ \} + \lambda\{\ ,\ \}_a$.

5. COMPLETENESS

By the completeness of a family $\mathcal{F}$ of commuting first integrals of a given Hamiltonian system $\dot{x} = \operatorname{sgrad} H(x)$ on a symplectic manifolds (M, ω) we mean that $\mathcal{F}$ contains $n = \frac{1}{2}\dim M$ functions which are independent on M almost everywhere. If M is not symplectic but Poisson, then for Liouville integrability one needs more integrals. Every Hamiltonian system, of course, possesses Casimir functions as standard integrals, but apart of them we need to find $\frac{1}{2}\operatorname{rank}\mathcal{A}$ non-trivial integrals so that the total number of independent first integrals is $\frac{1}{2}(\dim M + \operatorname{corank}\mathcal{A})$. Speaking more precisely, this condition guarantees the Liouville integrability on generic symplectic leaves. For our purposes it is convenient to use the following equivalent definition of completeness.

Definition 4. *A family $\mathcal{F}$ of commuting functions on a Poisson manifold $(M, \mathcal{A})$ is complete if the subspace $d\mathcal{F}(x) \subset T^*_x M$ generated by the differentials $df(x)$, $f \in \mathcal{F}$, is maximal isotropic w.r.t. $\mathcal{A}$ for almost all $x \in M$.*

In our bi-Hamiltonian situation, the completeness criterion for the family $\mathcal{F}_\mathcal{P}$ introduced in Proposition 1 is just a translation of Proposition 3 from Linear Algebra to Poisson Geometry.

Theorem 1. *The family $\mathcal{F}_\mathcal{P}$ is complete if and only if for a generic point $x \in M$ the following maximal rank condition holds*:

$$\operatorname{rank}\mathcal{A}_\lambda(x) = \operatorname{rank}\mathcal{P} \quad \textit{for all} \quad \lambda \in \overline{\mathbb{C}}. \tag{5.1}$$

As we know (see Proposition 1), the family $\mathcal{F}_\mathcal{P}$ is commutative w.r.t. any Poisson structure $\mathcal{A}_\lambda \in \mathcal{P}$. A priori, the completeness of $\mathcal{F}_\mathcal{P}$ (in the sense of Definition 4) might depend on the choice of $\mathcal{A}_\lambda$. So in order to be more precise, we should have specified a particular Poisson structure $\mathcal{A}_{\lambda_0}$ in the statement of Theorem 1. The conclusion of the theorem, however, says that the completeness condition does not depend on the particular choice of $\mathcal{A}_{\lambda_0}$. In other words, the statement of the theorem holds for any $\mathcal{A}_\lambda \in \mathcal{P}$.

There is a natural and efficient principle that allows us to verify the above necessary and sufficient completeness condition. To formulate it, we first notice that if the family $\mathcal{F}_\mathcal{P}$ is complete, then all the structures $\mathcal{A}_\lambda$ must be of the same rank on M, but, for each λ, the rank of $\mathcal{A}_\lambda(x)$ may drop on a certain singular set $S_\lambda = \{x \in M \mid \operatorname{rank}\mathcal{A}_\lambda(x) < \operatorname{rank}\mathcal{P}\}$. From the viewpoint of completeness these points are "bad". Condition (5.1) simply says that for completeness there must exist "good" points which belong to none of S_λ's. For such points to exist, it is sufficient to require that singular sets S_λ have codimension at least two. Then the union of sets S_λ over all λ's will have codimension at least one and its complement will consist of "good" points, as needed. This argument shows that a technically rather difficult problem of computing the rank of the Jacobi matrix (1.1) in the case of bi-Hamiltonian system can be reduced to the question about singular sets of the corresponding Poisson pencils.

In this *"codimension two principle"* there is a subtle point: we consider a real manifold M, but the parameter λ is complex so that from the real point of view, the "space of parameters" is not one-, but two-dimensional. However, in concrete examples we have to deal with, the difference between "real" and "complex", in fact, disappears. The point is that we usually work with algebraic objects (manifolds, Poisson structures, Casimir functions) which can be naturally complexified: we can introduce a new complex manifold $M^\mathbb{C}$ endowed with the complex Poisson pencil $\mathcal{P}^\mathbb{C}$ and construct the corresponding family of complex functions $\mathcal{F}^\mathbb{C}_\mathcal{P}$. In all natural situations, the complex functions that generate $\mathcal{F}^\mathbb{C}_\mathcal{P}$ are obtained from the real functions $f(x_1, \ldots, x_n)$ generating $\mathcal{F}_\mathcal{P}$ just by replacing real variables x_i with complex ones $z_i \in \mathbb{C}$. If such a complexification is well-defined, then we have the following

Codimension Two Principle. *Let all the brackets $\mathcal{A}_\lambda$, $\lambda \in \overline{\mathbb{C}}$, have the same rank and* $\operatorname{codim} S_\lambda \geqslant 2$ *for almost all $\lambda \in \overline{\mathbb{C}}$. Then $\mathcal{F}_\mathcal{P}^\mathbb{C}$ is complete. The completeness of $\mathcal{F}_\mathcal{P}^\mathbb{C}$ is equivalent to the completeness of $\mathcal{F}_\mathcal{P}$.*

We are going to illustrate now how this bi-Hamiltonian technology works in our two examples (argument shift method and n-dimensional rigid body dynamics).

Example A (Completeness). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $a \in \mathfrak{g}^*$ an arbitrary regular element. For the complexified Lie algebra $\mathfrak{g}^\mathbb{C}$, we consider the set of singular elements in its dual:

$$\operatorname{Sing} = \{x \in (\mathfrak{g}^\mathbb{C})^* \mid \operatorname{rank}\left(c_{ij}^k x_k\right) < \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}\} \subset (\mathfrak{g}^\mathbb{C})^*.$$

In terms of Poisson geometry, Sing is the set of singular points where the rank of the standard Lie–Poisson bracket is not maximal. From the algebraic point of view, Sing can be characterized as the union of singular coadjoint orbits (i.e., orbits of dimension smaller than $\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$).

Applying the Codimension Two Principle to the argument shift method immediately gives the following completeness criterion for $\mathcal{F}_a$ [2]: *the family of shifts $\mathcal{F}_a$ is complete on $\mathfrak{g}^*$ if and only if* $\operatorname{codim} \operatorname{Sing} \geqslant 2$.

As a simple corollary of this criterion we get the classical result by A.S. Mischenko and A.T. Fomenko, the original proof of which was absolutely different.

Theorem 1. *If $\mathfrak{g}$ is semisimple and $a \in \mathfrak{g}$ is regular, then the collection of commuting polynomials $\mathcal{F}_a$ is complete on $\mathfrak{g} \simeq \mathfrak{g}^*$. In other words, the basic shifts* (4.3) *are functionally independent on $\mathfrak{g}$.*

The "bi-Hamiltonian" proof consists in noticing that for semisimple complex Lie algebras $\operatorname{codim} \operatorname{Sing} = 3 \geqslant 2$.

Example B (Completeness). A similar result holds for Manakov integrals (4.5).

Theorem 1. *Let the eigenvalues of C be all distinct. Then the collection of commuting polynomials $\mathcal{F}_C$ is complete on* $\mathrm{so}(n)$.

Proof. According to the general bi-Hamiltonian construction we need to verify two conditions. The first condition is that the ranks of all the brackets $\{\ ,\ \}_{C+\lambda E}$ must be the same. This means that all the Lie algebras $\mathfrak{g}_\lambda$ given by the commutators $[\ ,\ ]_{C+\lambda E}$ on the space of skew-symmetric matrices must be of the same index. To verify this condition, we notice that if $\det(C + \lambda E) \neq 0$, then $\mathfrak{g}_\lambda$ is semisimple and its index is constant by continuity (if we change λ slightly, the algebra remains isomorphic to itself). Since the eigenvalues c_i of C are all distinct, in our family there are n non-semisimple Lie algebras that correspond to the combinations $C_i = C - c_i E$. The rank of the matrix C_i is $n - 1$, and it is not hard to see that the Lie algebra with the commutator $[\ ,\ ]_{C_i}$ is isomorphic to the semidirect sum $\mathrm{e}(n-1) = \mathrm{so}(n-1) +_\rho \mathbb{R}^{n-1}$, where ρ is the natural representation. It is well known that the index of $\mathrm{e}(n-1)$ is equal to $\left[\frac{n}{2}\right] = \operatorname{ind} \mathrm{so}(n)$, i. e., all brackets in the family are of the same rank.

The second condition is about the codimension of the set of singular elements for a generic bracket $\{\ ,\ \}_{C+\lambda E}$. But generic brackets are semisimple so that this codimension is $3 \geqslant 2$, as needed.

We emphasize once again that from the theory of Lie algebras in this statement we only used the fact about the codimension, the rest follows immediately from the bi-Hamiltonian interpretation of the Jordan–Kronecker theorem. More precisely, the bi-Hamiltonian approach allows us to reformulate the facts we are interested in (in the present case, this is the functional independence of a rather complicated collection of polynomials) into an absolutely different language which essentially simplifies the verification. The further content of our paper is just an illustration of this idea. The bi-Hamiltonian approach serves here as a bridge between two equivalent formulations.

6. SET OF SINGULARITIES

As was already noticed, the structure of the set of critical points of the momentum mapping plays an important role in the study of topological properties of integrable Hamiltonian systems.

The standard situation is that we have n commuting independent (almost everywhere) integrals $F_1, \dots, F_n : M \to \mathbb{R}$ on a symplectic manifold (M^{2n}, ω), and we want to find the set of those points where the rank of the corresponding Jacobi matrix is less than n.

In real problems we have to deal with, the situation may often be slightly different. That is why we define the set of critical points in more general terms.

First of all, the phase space of a system is often not symplectic, but Poisson. In this case, it is natural to add Casimir functions to a given family $\mathcal{F}$ of first integrals and consider them all together. Also, for some families $\mathcal{F}$ there is no canonical method for choosing a basis. To avoid this ambiguity, it is convenient to work with the Poisson algebra generated by the given commuting integrals and Casimir functions. Since we do not add any essentially new integrals, we will use the same notation $\mathcal{F}$ for this "wider" algebra of first integrals.

Definition 5. *Let $(M, \mathcal{A})$ be a Poisson manifold and let $\mathcal{F} \subset C^\infty(M)$ be a commutative Poisson algebra of functions on M which is complete in the sense of Definition 4. We will say that a point $x \in M$ is a critical point for $\mathcal{F}$ if the subspace $d\mathcal{F}(x) \subset T_x^*M$ generated by the differentials $df(x)$ of all functions $f \in \mathcal{F}$ is not maximal isotropic with respect to $\mathcal{A}$.*

It is clear that the standard definition of a critical point of the momentum mapping is a particular case of Definition 5. The reason for such a modification is that now we don't need to fix any universal basis in the algebra of integrals, but may chose appropriate basis integrals depending on a point $x \in M$ under consideration which can be quite convenient.

It follows from this definition that those points at which the rank of the Poisson structure drops are automatically critical.

We now consider a bi-Hamiltonian dynamical system and take the algebra $\mathcal{F}_\mathcal{P}$ of its integrals generated by the Casimir functions of the pencil of Poisson brackets $\mathcal{P} = \{\mathcal{A} + \lambda\mathcal{B} \mid \lambda \in \mathbb{R}\}$. Suppose that this algebra is complete and therefore according to our general construction, all the brackets in the pencil are of the same rank.

Our goal is to find the critical points of $\mathcal{F}_\mathcal{P}$. Under the natural assumptions (see Section 3), the *set of critical points for the family $\mathcal{F}_\mathcal{P}$*

$$K_\mathcal{P} = \left\{x \in M \;\middle|\; \dim d\mathcal{F}_\mathcal{P}(x) < \tfrac{1}{2}(\dim M + \operatorname{corank}\mathcal{P})\right\}$$

can be described as follows.

As before, for each $\lambda \in \mathbb{C}$, we consider the *set of singular points of the Poisson structure $\mathcal{A}_\lambda$* in M

$$S_\lambda = \{x \in M \mid \operatorname{rank}(\mathcal{A}(x) + \lambda\mathcal{B}(x)) < \operatorname{rank}\mathcal{P}\}.$$

In addition, we formally set $S_\infty = \{x \in M \mid \operatorname{rank}\mathcal{B}(x) < \operatorname{rank}\mathcal{P}\}$. Also consider the *set of singular points of the pencil $\mathcal{P}$*:

$$S_\mathcal{P} = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_\lambda.$$

Theorem 2. *A point $x \in M$ is a critical point for $\mathcal{F}_\mathcal{P}$ if and only if there exists $\lambda \in \overline{\mathbb{C}}$ such that $x \in S_\lambda$. In other words, $K_\mathcal{P} = S_\mathcal{P}$.*

Since even in the real situation we have to consider complex values of the parameter λ, it is often convenient to complexify all the objects from the very beginning. If we do so, then the set $S_\lambda^\mathbb{C}$ of singular points of the complex Poisson bracket $\mathcal{A}_\lambda^\mathbb{C}$ can be considered as a subset in the complexified Poisson manifold $M^\mathbb{C}$. Then the set of singular points of the (complexified) pencil $\mathcal{P}$

$$S_\mathcal{P}^\mathbb{C} = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_\lambda^\mathbb{C}$$

is the set $K^{\mathbb{C}}_{\mathcal{P}}$ of critical points for the family of complex functions $\mathcal{F}^{\mathbb{C}}_{\mathcal{P}}$ on the complex manifold $M^{\mathbb{C}}$, and to turn back to the real case, we just need to take the real part of it:

$$K_{\mathcal{P}} = \mathrm{Re}(K^{\mathbb{C}}_{\mathcal{P}}) = K^{\mathbb{C}}_{\mathcal{P}} \cap M.$$

This point of view is natural because $S^{\mathbb{C}}_{\lambda} \subset M^{\mathbb{C}}$ usually has a simpler geometric structure than its real part $S_{\lambda} = \mathrm{Re}(S^{\mathbb{C}}_{\lambda}) = S^{\mathbb{C}}_{\lambda} \cap M$.

Thus, in the case of bi-Hamiltonian systems the set of critical points of the momentum mapping has a natural description in terms of the singular sets of the Poisson structures $\mathcal{A}_{\lambda}$, $\lambda \in \overline{\mathbb{C}}$. Here we see the following general principle: the singularities of the Lagrangian fibration associated with a bi-Hamiltonian system are defined by the singularities of the pencil $\mathcal{P}$.

Example A (Singularities). Let $\mathfrak{g} = \mathfrak{g}^*$ be a real semisimple Lie algebra identified with its dual space by means of the Killing form. Let $\mathrm{Sing} \subset \mathfrak{g}^{\mathbb{C}}$ be the set of singular points in the sense of the adjoint representation and $a \in \mathfrak{g}$ a regular element. The next theorem describes the set of critical points K_a for the commutative collection of polynomials $\mathcal{F}_a$ obtained by the argument shift method.

Theorem 2. *An element $x \in \mathfrak{g}$ is a critical point of $\mathcal{F}_a$ (or, equivalently, of the momentum mapping Φ_a) if and only if there exists $\lambda \in \mathbb{C}$ such that $x + \lambda a$ is a singular element in $\mathfrak{g}^{\mathbb{C}}$.*

In other words, the set of critical points K_a of $\mathcal{F}_a$ is the (real part of the) cylinder over the set of singular elements Sing *with the generating line parallel to a, that is*:

$$K_a = (\mathrm{Sing} + \mathbb{C} \cdot a) \cap \mathfrak{g}.$$

Remark. The same result holds for any finite-dimensional Lie algebra: $K_a = (\mathrm{Sing} + \mathbb{C} \cdot a) \cap \mathfrak{g}^*$, where $a \in \mathfrak{g}^*$ is regular and Sing denotes the set of singular elements in $(\mathfrak{g}^{\mathbb{C}})^*$. In particular, if $\operatorname{codim} \mathrm{Sing} = 1$, then K_a coincides with the whole coalgebra g^*, which means that all points are critical and, therefore, the family $\mathcal{F}_a$ is not complete.

Notice that the description of the singular set Sing for a semisimple Lie algebra is quite simple. For instance, in the case of $\mathrm{sl}(n, \mathbb{C})$ the regularity condition can be formulated as follows: an $(n \times n)$-matrix $x \in \mathrm{sl}(n, \mathbb{C})$ is regular if and only if for each its eigenvalue there is exactly one eigenvector. Equivalently, x is regular if and only if its minimal polynomial coincides with the characteristic one. Correspondingly, $x \in \mathrm{sl}(n, \mathbb{C})$ is singular if x has an eigenvalue with at least two linearly independent eigenvectors.

Example B (Singularities). We now want to describe the set K_C of skew-symmetric matrices $X \in \mathrm{so}(n)$ where the differentials of the Manakov integrals (4.5) are linearly dependent or, which is the same, the set of critical points for the momentum mapping (4.6):

$$\Phi_C : \mathrm{so}(n) \to \mathbb{R}^s, \qquad s = \frac{1}{2}(\dim \mathrm{so}(n) + \operatorname{ind} \mathrm{so}(n)).$$

The bi-Hamiltonian answer to this question is very simple:

Theorem 2. *A skew-symmetric $(n \times n)$-matrix X is a critical point of $\mathcal{F}_C$ (or, equivalently, of the momentum mapping Φ_C) if and only if there exists $\lambda \in \overline{\mathbb{C}}$ such that $X \in S_{\lambda}$, where $S_{\lambda} \subset \mathrm{so}(n)$ is the singular set of the bracket $\{\ ,\ \}_{C+\lambda E}$. In other words,*

$$K_C = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_{\lambda}.$$

Since the Lie algebras $\mathfrak{g}_{\lambda}$ (skew-symmetric matrices with the commutator $[\ ,\ ]_{C+\lambda E}$) are almost all isomorphic (after complexification), we are able to describe the singular set for each of them by using an appropriate isomorphism with the "model Lie algebra" $\mathrm{so}(n, \mathbb{C})$. If $C + \lambda E$ is non-degenerate, then such an isomorphism can be defined by

$$X \mapsto (C + \lambda E)^{1/2} X (C + \lambda E)^{1/2},$$

where $(C+\lambda E)^{1/2}$ is a symmetric matrix (in general, complex). Using this observation, it is easy to see that the singular set S_λ for the bracket $\{\ ,\ \}_{C+\lambda E}$ can be described as

$$S_\lambda = \mathrm{Re}\big((C+\lambda E)^{1/2}\ \mathrm{Sing}\,(C+\lambda E)^{1/2}\big),$$

where $\mathrm{Sing} \subset \mathrm{so}(n,\mathbb{C})$ is the union of all singular orbits (algebraic variety of codimension 3).

Thus, the statement of Theorem 2B can be reformulated as follows: *A skew-symmetric $(n\times n)$-matrix X is a critical point of $\mathcal{F}_C$ (or, equivalently, of the momentum mapping Φ_C) if and only if one of the two condition holds*:

1) *there exists $\lambda \in \overline{\mathbb{C}}$ such that $\det(C+\lambda E)\neq 0$ and the skew-symmetric matrix $(C+\lambda E)^{-1/2}\times X(C+\lambda E)^{-1/2}$ is a singular element in $\mathrm{so}(n,\mathbb{C})$;*
2) *X is a singular covector in the sense of the Lie algebra defined on the space of skew-symmetric matrices by the commutator $[\ ,\ ]_{C-c_iE}$, where $c_i\in\mathbb{R}$ is one of the eigenvalues of C.*

In the case of dimension 4, this description becomes much clearer if we notice that the set Sing is just the union of two transversally intersecting 3-dimensional subspaces (these subspaces are exactly the components of the standard decomposition $\mathrm{so}(4)=\mathrm{so}(3)\oplus\mathrm{so}(3)$): namely, $\mathrm{Sing}\subset\mathrm{so}(4,\mathbb{C})$ is the union of two 3-dimensional subspaces

$$P_1 = \begin{pmatrix} 0 & z_3 & -z_2 & z_1\\ -z_3 & 0 & z_1 & z_2\\ z_2 & -z_1 & 0 & z_3\\ -z_1 & -z_2 & -z_3 & 0\end{pmatrix} \qquad \text{and} \qquad P_2 = \begin{pmatrix} 0 & -z_3 & z_2 & z_1\\ z_3 & 0 & -z_1 & z_2\\ -z_2 & z_1 & 0 & z_3\\ -z_1 & -z_2 & -z_3 & 0\end{pmatrix}.$$

Hence, the set of critical points of the Manakov integrals $\mathcal{F}_C$ is

$$K_C = \bigcup_{\lambda\in\overline{\mathbb{C}}} (P_1^\lambda \cup P_2^\lambda),$$

where $P_i^\lambda = \mathrm{Re}\big((C+\lambda E)^{1/2}\,P_i\,(C+\lambda E)^{1/2}\big)$. Thus, in the case of dimension 4, we obtain a very natural parametrization of the set of critical points K_C by means of 4 parameters z_1, z_2, z_3, λ.

7. EQUILIBRIUM POINTS

As before, $\mathcal{P}=\{\mathcal{A}_\lambda\}$ denotes a pencil of compatible Poisson structures on a manifold M, and $\mathcal{F}_\mathcal{P}$ is the corresponding algebra generated by the Casimir functions of generic brackets $\mathcal{A}_\mu\in\mathcal{P}$ (which is assumed to be complete on M).

Let x belong to a regular symplectic leaf of the Poisson structure $\mathcal{A}=\mathcal{A}_0$. We say that x is a *common equilibrium point for $\mathcal{F}_\mathcal{P}$* if $\mathrm{sgrad}_\mathcal{A} f(x)=0$ for any $f\in\mathcal{F}_\mathcal{P}$, where $\mathrm{sgrad}_\mathcal{A} f(x)$ is, by definition, the Hamiltonian vector field related to $f(x)$, i.e., $\mathcal{A}{\cdot}df(x)$.

Theorem 3. *A point $x\in M$ is a common equilibrium point for $\mathcal{F}_\mathcal{P}$ if and only if the kernels $\mathrm{Ker}\,\mathcal{A}_\mu(x)$ of all brackets which are generic at this point coincide.*

Equivalently, for x to be a common equilibrium it is sufficient to require that the kernels of *just two* brackets (generic at this point) coincide: $\mathrm{Ker}\,\mathcal{A}_\mu(x)=\mathrm{Ker}\,\mathcal{A}_{\mu'}(x)$, $\mu\neq\mu'$.

Proof is obvious. If all the kernels coincide, then for any $f\in\mathcal{Z}(\mathcal{A}_\mu)$ we have $df(x)\in\mathrm{Ker}\,\mathcal{A}_\mu(x)=\mathrm{Ker}\,\mathcal{A}(x)$ and, therefore, $\mathrm{sgrad}_\mathcal{A} f(x)=0$. Since $\mathcal{F}_\mathcal{P}$ is generated by the Casimir functions of generic brackets, we have $\mathrm{sgrad}_\mathcal{A} f(x)=0$ for any $f\in\mathcal{F}_\mathcal{P}$. Conversely, if x is a common equilibrium point for $\mathcal{F}_\mathcal{P}$, then $df(x)\in\mathrm{Ker}\,\mathcal{A}(x)$ for any $f\in\mathcal{F}_\mathcal{P}$, i.e., $\mathrm{span}\{df(x)\mid f\in\mathcal{F}_\mathcal{P}\}=\mathrm{Ker}\,\mathcal{A}(x)$. But $\mathcal{F}_\mathcal{P}$ is generated by the Casimir functions, so that the span of all $df(x)$'s contains all (regular) kernels. Hence, $\mathrm{Ker}\,\mathcal{A}_\lambda(x)=\mathrm{Ker}\,\mathcal{A}(x)$ for each (generic) λ.

It is interesting to notice that the bi-Hamiltonian nature of the family $\mathcal{F}_\mathcal{P}$ implies that being a common equilibrium in the sense of a particular Poisson structure $\mathcal{A}_{\lambda_0}\in\mathcal{P}$, the point x will be a common equilibrium for every $\mathcal{A}_\lambda\in\mathcal{P}$.

Remark. Notice that the study of common equilibria may look a bit artificial problem because we are usually interested in equilibrium points for a specific Hamiltonian rather than for all first integrals. However, this setting does not restrict the generality much: the most natural, namely, *isolated* equilibrium points are always common. Non-isolated equilibria, of course, appear in integrable systems too, but they have different nature and should be treated separately.

Theorem 3 makes finding common equilibria almost trivial. We shall demonstrate this for our two models.

Example A (Equilibria). We would like to describe those points $x \in \mathfrak{g} = \mathfrak{g}^*$ for which $\operatorname{sgrad} f(x) = 0$ for any $f \in \mathcal{F}_a$. To that end, we simply need to compare two objects: the kernel of the standard Lie–Poisson bracket $\{\ ,\ \}$ and the kernel of the "frozen" bracket $\{\ ,\ \}_a$ at x.

In the semisimple case, these kernels are respectively the centralizers of x and of a in $\mathfrak{g}$. If a is semisimple and regular, then its centralizer is a Cartan subalgebra $\mathfrak{h}_a$. It is easy to see that the centralizer of x is the same if and only if $x \in \mathfrak{h}_a$. Thus, we have

Theorem 3. *A point x is a common equilibrium point for $\mathcal{F}_a$ if and only if $x \in \mathfrak{h}_a$.*

Remark. A similar statement obviously holds for an arbitrary element $a \in \mathfrak{g}$ (not necessarily regular and semisimple); instead of the Cartan subalgebra $\mathfrak{h}_a$ we simply take the centralizer of a. Moreover, this statement can naturally be generalized to the case of an arbitrary finite-dimensional Lie algebra: $x \in \mathfrak{g}^*$ is a common equilibrium point for $\mathcal{F}_a$ (for $a \in \mathfrak{g}^*$ being regular) if and only if $\operatorname{Ann}(a) \subset \operatorname{Ann}(x)$.

Example B (Equilibria). The common equilibrium points for the Manakov integrals can be found in a similar way. We shall assume here for simplicity that $X \in \mathrm{so}(n)$ is a regular element.

According to Theorem 3, X is a common equilibrium point if and only if the kernel of $\{\ ,\ \} = \{\ ,\ \}_E$ coincides with (more precisely, is contained in) the kernel of $\{\ ,\ \}_C$.

The kernel of $\{\ ,\ \}$ at the point X is a Cartan subalgebra $\mathfrak{h}_X \subset \mathrm{so}(n)$ generated by X (or equivalently, the centralizer of X). It is described by the equation

$$XY - YX = 0. \tag{7.1}$$

The kernel of $\{\ ,\ \}_C$ is defined as the solution to the following matrix equation:

$$CYX - XYC = 0, \tag{7.2}$$

where Y is unknown.

It is not hard to describe those X's for which the solution sets of (7.1) and (7.2) are the same. The answer is almost obvious. But since such a situation (the so-called Lie pencil) is quite interesting in a more general setting, we would like to suggest the following idea.

The kernel of any bracket $\{\ ,\ \}_{C+\lambda E}$ from our pencil at a given point X is a subalgebra w.r.t. $[\ ,\ ]_{C+\lambda E}$. Moreover, for generic λ and X this subalgebra is commutative.

We want to describe the case when all these subalgebras coincide for generic λ's. In particular, we see that this subalgebra is commutative w.r.t. to all brackets simultaneously, which is a very strong restriction.

In our case this condition means that the Cartan subalgebra $\mathfrak{h}_X$ generated by X is commutative with respect to the commutator $[\ ,\ ]_C$ (and therefore, automatically with respect to any commutator $[\ ,\ ]_{C+\lambda E}$). The description of the Cartan subalgebras satisfying this property leads to the following

Theorem 3. *A matrix $X \in \mathrm{so}(n)$ is a common equilibrium point for $\mathcal{F}_C$ if and only if there exists an orthonormal basis with respect to which C is diagonal and X has the standard block diagonal form*:

$$X = \begin{pmatrix} 0 & x_{12} & & & \\ -x_{12} & 0 & & & \\ & & 0 & x_{34} & \\ & & -x_{34} & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Equivalently, the set of common equilibrium points for $\mathcal{F}_C$ (with C diagonal) is the union of the standard block-diagonal Cartan subalgebra

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} 0 & x_{12} & & & \\ -x_{12} & 0 & & & \\ & & 0 & x_{34} & \\ & & -x_{34} & 0 & \\ & & & & \ddots \end{pmatrix}, \quad x_{i,i+1} \in \mathbb{R} \right\}$$

and all the subalgebras obtained from $\mathfrak{h}_0$ by conjugations $\mathfrak{h} \mapsto P\mathfrak{h}P^{-1}$, where P is a permutation matrix.

These two results admits the following more general interpretation. Assume that we have a Lie pencil $\mathcal{P} = \{[\,,\,]_\lambda,\ \lambda \in \mathbb{R}\}$ defined on a semisimple Lie algebra $\mathfrak{g}$. Following the standard procedure, we can construct a commutative family of polynomials $\mathcal{F}_\mathcal{P}$ on $\mathfrak{g}^* = \mathfrak{g}$ and then ask the question about common equilibria for $\mathcal{F}_\mathcal{P}$.

Proposition 4. *The set of common equilibrium points is the union of Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ which can be characterized by the following property: $\mathfrak{h}$ is a common Cartan subalgebra for all $\mathfrak{g}_\lambda$ corresponding to "generic" $\lambda \in \mathbb{C}$.*

8. NON-DEGENERACY CONDITION FOR EQUILIBRIUM POINTS

The next problem is to verify the non-degeneracy condition and to determine the types of equilibrium points.

Let x be a common equilibrium point for $\mathcal{F}_\mathcal{P}$. As has just been shown, this means that we have the following condition: all generic kernels of $\mathcal{A}_\lambda(x)$ are the same and we shall denote this common kernel by $Z \subset T_x^*M$. In particular, this means that after taking quotient with respect to Z we obtain a non-degenerate pencil of skew-symmetric forms $A_\lambda = A + \lambda B$ on T_x^*M/Z. Notice that T_x^*M/Z can naturally be considered as the dual to the tangent space $T_x\mathcal{O}$ to the symplectic leaf $\mathcal{O}$ through x (recall that at a common equilibrium point the tangent space $T_x\mathcal{O}$ does not depend on the choice of a generic Poisson structure $\mathcal{A}_\mu \in \mathcal{P}$).

Consider a Hamiltonian vector field $\operatorname{sgrad}_\mathcal{A} f$, where $\mathcal{A} \in \mathcal{P}$ and $f \in \mathcal{F}_\mathcal{P}$. It vanishes at $x \in M$, and we may consider its linear part as the linear operator $\mathcal{A}{\cdot}d^2f(x)$ acting on the tangent space T_xM. Clearly, $T_x\mathcal{O}$ is an invariant subspace for such operator, and we shall denote the corresponding restriction by $P_f : T_x\mathcal{O} \to T_x\mathcal{O}$. The operators P_f belong to the symplectic Lie algebra $\operatorname{sp}(T_x\mathcal{O}, A)$ and generate a commutative subalgebra $\mathfrak{k}$ in it.

The bi-Hamiltonian structure comes into play if we notice that $\mathcal{F}_\mathcal{P}$ is generated by functions $f \in \mathcal{F}_\mathcal{P}$ for each of which we can find another function $f' \in \mathcal{F}_\mathcal{P}$ satisfying

$$A{\cdot}df = B{\cdot}df'. \tag{8.1}$$

Since A and B are both well-defined and non-degenerate on $T_x^*\mathcal{O} = T_x^*M/Z$, we can naturally define the usual recursion operator $Q = BA^{-1} : T_x\mathcal{O} \to T_x\mathcal{O}$. It is easy to see that (8.1) gives $QP_f = P_{f'}$ for the corresponding linearizations.

This immediately implies

Proposition 5. *Each $P_f \in \operatorname{sp}(T_x\mathcal{O}, A)$ belongs at the same time to the symplectic Lie algebra $\operatorname{sp}(T_x\mathcal{O}, B)$. In particular, $\mathfrak{k} \subset \operatorname{sp}(T_x\mathcal{O}, A) \cap \operatorname{sp}(T_x\mathcal{O}, B)$. Moreover, this subalgebra $\mathfrak{k}$ is invariant with respect to the recursion operator Q in the sense that for any element $P \in \mathfrak{k}$ we have $QP \in \mathfrak{k}$.*

According to Definition 1, the non-degeneracy of x means that $\mathfrak{k}$ is a Cartan subalgebra in $\mathrm{sp}(T_x\mathcal{O}, A)$ (and in $\mathrm{sp}(T_x\mathcal{O}, B)$ too!). It is not hard to see that this condition imposes rather strong (purely algebraic) restrictions on the triple $(\mathfrak{k}, A, B)$.

Below we give two (abstract) algebraic statements which describe some properties of $(\mathfrak{k}, A, B)$, where A and B are non-degenerate skew-symmetric bi-linear forms on a vector space V and $\mathfrak{k}$ is a common commutative subalgebra in $\mathrm{sp}(V, A)$ and $\mathrm{sp}(V, B)$ (we may think of V as the tangent space $T_x\mathcal{O}$).

Proposition 6. *Let $\mathfrak{k}$ be a common Cartan subalgebra for two symplectic Lie algebras $\mathrm{sp}(V, A)$ and $\mathrm{sp}(V, B)$. Then there exists a (complex) basis in $V^{\mathbb{C}}$ such that the pencil $A + \lambda B$ and the subalgebra $\mathfrak{k}$ can be simultaneously "diagonalized", that is*:

$$A+\lambda B = \begin{pmatrix} 0 & \lambda_1+\lambda & & & & \\ -\lambda_1-\lambda & 0 & & & & \\ & & 0 & \lambda_2+\lambda & & \\ & & -\lambda_2-\lambda & 0 & & \\ & & & & \ddots & \end{pmatrix}, \qquad \mathfrak{k} = \left\{ \begin{pmatrix} h_1 & 0 & & & & \\ 0 & -h_1 & & & & \\ & & h_2 & 0 & & \\ & & 0 & -h_2 & & \\ & & & & \ddots & \end{pmatrix} \right\}$$

(*where some of λ_i's may coincide*).

The "converse" statement is

Proposition 7. *Let $\mathfrak{k}$ be a common commutative subalgebra in $\mathrm{sp}(V, A)$ and $\mathrm{sp}(V, B)$ which is invariant with respect to the recursion operator $Q = AB^{-1} : V \to V$ (i.e., for any $P \in \mathfrak{k}$ we have $QP \in \mathfrak{k}$). If the "characteristic polynomial" $\det(A + \lambda B)$ has only roots of multiplicity 2 and $\mathfrak{k}$ contains a non-degenerate operator P (i.e., $\det P \neq 0$), then $\mathfrak{k}$ is a common Cartan subalgebra in $\mathrm{sp}(V, A)$ and $\mathrm{sp}(V, B)$.*

Coming back to our discussion about the non-degeneracy condition, we can reformulate the above algebraic statements as follows:

Theorem 4. *Let x be a common equilibrium point for $\mathcal{F}_{\mathcal{P}}$.*

1) *If x is non-degenerate, then the Jordan–Kronecker decomposition for the pencil $\mathcal{A}_\lambda(x) = \mathcal{A}(x) + \lambda\mathcal{B}(x)$ at the point x consists of one trivial $(r \times r)$-block (corresponding to the common kernel $Z = \mathrm{Ker}\,\mathcal{A}(x) = \mathrm{Ker}\,\mathcal{B}(x)$) and (2×2)-blocks of the form*

$$\begin{pmatrix} 0 & \lambda_i + \lambda \\ -\lambda_i - \lambda & 0 \end{pmatrix}, \quad i = 1, \ldots, \frac{1}{2}(\dim M - \mathrm{corank}\,\mathcal{A}).$$

In other words, the pencil is diagonalizable (over $\mathbb{C}$).

2) *Suppose that the rank of $\mathcal{A}_\lambda(x) = \mathcal{A}(x) + \lambda\mathcal{B}(x)$ drops for $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$, $q = \frac{1}{2}(\dim M - \mathrm{corank}\,A)$, where all λ_i's are distinct, and there exists $f \in \mathcal{F}_{\mathcal{P}}$ such that the corresponding linearization $P_f : T_x\mathcal{O} \to T_x\mathcal{O}$ has no zero eigenvalues (or, equivalently, $\det P_f \neq 0$). Then x is non-degenerate.*

In fact, this theorem says that the Jordan–Kronecker decomposition (in the case of a nondegenerate singularity) is essentially the same as the decomposition given by the Eliasson theorem (see Section 1). Now we are going to see how (the second part of) Theorem 4 works in our two examples.

Example A (Non-Degeneracy). Let $\mathfrak{h}_a$ be the Cartan subalgebra generated by the element $a \in \mathfrak{g}^* = \mathfrak{g}$ and $x \in \mathfrak{h}_a$. As we know, x is a common equilibrium point for $\mathcal{F}_a$. The following statement gives sufficient non-degeneracy conditions for x.

Theorem 4. *Let $\alpha_1, \dots, \alpha_q$ be the positive roots associated with the complexification $\mathfrak{h}_a^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, $q = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})$. Consider the collection of numbers*

$$\lambda_i = \frac{\alpha_i(x)}{\alpha_i(a)}, \qquad i = 1, \dots, q.$$

If all these numbers are distinct, then x is a non-degenerate equilibrium point.

Proof. The rank of the linear combination $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ at the point x is not maximal if and only if $x + \lambda a$ is a singular element of $\mathfrak{g}^{\mathbb{C}}$. Since $x + \lambda a$ belongs to the Cartan subalgebra $\mathfrak{h}_a^{\mathbb{C}}$, "being singular" for this element means that there exists a root α_i such that $\alpha_i(x + \lambda a) = 0$ or, equivalently $\lambda = -\lambda_i = -\dfrac{\alpha_i(x)}{\alpha_i(a)}$. Thus, the condition of our theorem is equivalent to the fact that the rank of $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ at the equilibrium point x drops for q distinct values of λ (geometrically this means that the affine line $x + \lambda a$ is in a generic position w.r.t. the Weyl chambers).

According to Theorem 4, to complete the proof we only need to find a function $f \in \mathcal{F}_a$ for which the linear part of the corresponding Hamiltonian vector field $\operatorname{sgrad} f(y) = [y, df(y)]$ at the point x would be non-degenerate on the tangent space to the adjoint orbit $\mathcal{O}_x$. First of all, we notice that $T_x\mathcal{O}_x$ is the orthogonal complement to the Cartan subalgebra $\mathfrak{h}_x$ generated by x, but $\mathfrak{h}_a = \mathfrak{h}_x$, i.e., $T_x\mathcal{O}_x = \mathfrak{h}_a^{\perp}$.

Now let $f(y) = \langle a, y\rangle$ where $\langle\ ,\ \rangle$ denotes the Killing form. This is a Casimir function of $\{\ ,\ \}_a$ and, therefore, $f \in \mathcal{F}_a$. The vector field $\operatorname{sgrad} f(y) = [y, a] = -\operatorname{ad}_a y$ is linear and the kernel of $-\operatorname{ad}_a$ coincides with the Cartan subalgebra $\mathfrak{h}_a$ so that the restriction of $-\operatorname{ad}_a$ onto $\mathfrak{h}_a^{\perp} = T_x\mathcal{O}_x$ is non-degenerate, as required.

Example B (Non-Degeneracy). For simplicity we consider the even-dimensional case $\mathrm{so}(2n)$. Let

$$X = \begin{pmatrix} 0 & x_{12} & & & \\ -x_{12} & 0 & & & \\ & & 0 & x_{34} & \\ & & -x_{34} & 0 & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 & & & & \\ & c_2 & & & \\ & & c_3 & & \\ & & & c_4 & \\ & & & & \ddots \end{pmatrix}.$$

For each pair $x_{i,i+1}$, $x_{j,j+1}$, we consider the following quadratic equation

$$\frac{x_{i,i+1}^2}{x_{j,j+1}^2} = \frac{(c_i + \lambda)(c_{i+1} + \lambda)}{(c_j + \lambda)(c_{j+1} + \lambda)} \tag{8.2}$$

in λ. Let λ_{ij} and λ'_{ij} be its roots.

Theorem 4. *If $\lambda_{ij}, \lambda'_{ij}$ $(i \neq j,\ \ i, j = 1, 3, \dots, 2n-1)$ are all distinct, then X is a non-degenerate equilibrium point of $\mathcal{F}_C$.*

Proof. We use the same idea. First of all, we would like to describe those values of λ for which the rank of $\{\ ,\ \}_{C+\lambda E}$ drops at the point X or, equivalently, X is a singular point in the sense of the Lie algebra $\mathfrak{g}_\lambda$ (defined by the commutator $[\ ,\ ]_{C+\lambda E}$ on the space of skew-symmetric matrices). Since $\mathfrak{g}_\lambda$ is isomorphic to $\mathrm{so}(n, \mathbb{C})$ for $\lambda \neq -c_i$, we may use the corresponding isomorphism to see

that X is singular for $\mathfrak{g}_\lambda$ if and only if the matrix

$$X' = (C+\lambda E)^{-\frac{1}{2}} X (C+\lambda E)^{-\frac{1}{2}} = \begin{pmatrix} 0 & x'_{12} & & & \\ -x'_{12} & 0 & & & \\ & & 0 & x'_{34} & \\ & & -x'_{34} & 0 & \\ & & & & \ddots \end{pmatrix},$$

where $x'_{i,i+1} = \dfrac{x_{i,i+1}}{\sqrt{(c_i+\lambda)(c_{i+1}+\lambda)}}$, is singular for the standard $\mathrm{so}(n,\mathbb{C})$; see Section 6. The latter is equivalent to the relation

$$x'^2_{i,i+1} = x'^2_{j,j+1},$$

which is exactly (8.2). Thus, the solutions λ_{ij}, λ'_{ij} of (8.2) are exactly the characteristic numbers of the pencil $\{\ ,\ \}_{C+\lambda E}$ at the equilibrium point X.

All these numbers λ_{ij}, λ'_{ij} are distinct. It remains to apply Theorem 4, according to which we need to find a function $f \in \mathcal{F}_C$ such that the corresponding linearization P_f is non-degenerate (on the tangent space to the orbit $\mathcal{O}_X$).

We take $f(Y) = \operatorname{Tr} CY^2 \in \mathcal{F}_C$. The corresponding Hamiltonian equation is

$$\dot{Y} = [df(Y), Y] = [CY + YC, Y] = [C, Y^2].$$

The linearized equation at the point X is

$$\dot{Y} = [C, YX + XY]. \tag{8.3}$$

We need to show that the operator $Y \mapsto [C, YX + XY]$ is non-degenerate on $T_X\mathcal{O}_X$. But this fact is almost obvious. Indeed, since C is regular, the kernel of this operator consists of those Y for which $YX + XY$ is diagonal. But this happens if and only if Y belongs to the Cartan subalgebra $\mathfrak{h}_X$ generated by X. Clearly $\mathfrak{h}_X \cap T_X\mathcal{O}_X = \{0\}$, so the linearized system (8.3) is non-degenerate on $T_X\mathcal{O}_X$, as needed. This completes the proof.

9. CORANK 1 SINGULARITIES

Assume that $x \in M$ is a critical point of corank 1. This means that the dimension of the subspace $d\mathcal{F}_\mathcal{P}(x)$ is $k-1$, where k is the dimension of the maximal isotropic subspace in T^*_xM. It follows from the Jordan–Kronecker decomposition theorem that there exists a unique $\lambda \in \mathbb{R}$ such that the rank of $\mathcal{A}(x) + \lambda\mathcal{B}(x)$ is not maximal. Moreover, the multiplicity of λ is minimal, i.e., equals 2. In other words, this characteristic number λ gives a single (2×2) Jordan block in the Jordan–Kronecker decomposition and $\dim \operatorname{Ker} \mathcal{A}_\lambda(x) = \operatorname{corank} \mathcal{P} + 2$.

It turns out that the structure of the singularity at the point x is determined by two things:

- the singularity of the Poisson structure $\mathcal{A}_\lambda$ at this point;
- the behavior of the other Casimir functions with respect to this singularity.

According to the Weinstein theorem [30], the Poisson structure $\mathcal{A}_\lambda$ in a small neighborhood of x splits into direct product of the transversal Poisson structure and the non-degenerate Poisson structure defined on the symplectic leaf through x. For our purposes we need to understand the properties of the transversal structure. In general, we cannot say much about it. That is why in this section from the very beginning we shall assume that the Poisson structure $\mathcal{A}_\lambda$ is semisimple in the sense that M has a natural identification with a real semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}^*$ endowed with the standard Lie–Poisson bracket.

Then x is a singular element in the semisimple Lie algebra $\mathfrak{g}$ and the dimension of its centralizer (which, as we know, coincides with the kernel of $\mathcal{A}_\lambda(x)$) is $\operatorname{ind}\mathfrak{g} + 2$. For simplicity, we shall assume that $x \in \mathfrak{g}$ is a semisimple element (this is obviously a generic case). Then the centralizer of x in $\mathfrak{g}$ is a Lie subalgebra of the form $\mathfrak{u} \oplus \mathbb{R}^{l-1}$, where $\mathfrak{u}$ is a three-dimensional real semisimple Lie algebra and $l = \operatorname{ind}\mathfrak{g}$.

Without loss of generality in a neighborhood of x, we may assume that the Casimir functions of $\mathcal{A}_\lambda$ are some of basis functions from $\mathcal{F}_\mathcal{P}$ that define the structure of the Lagrangian fibration in a neighborhood of x. It is easy to see that these functions can be naturally identified with the coadjoint invariants of the centralizer $\mathfrak{u} \oplus \mathbb{R}^{l-1}$. One can show (see [2]) that in a neighborhood of x we can chose a local coordinate system $z_1, \dots, z_{l-1},\ u_1, u_2, u_3,\ p_1, \dots, p_s, q_1, \dots, q_s$ in such a way that

- $p_1, \dots, p_s, q_1, \dots, q_s$ are symplectic coordinates on the singular orbit passing through x and $z_1, \dots, z_{l-1}, u_1, u_2, u_3$ are coordinates on the transversal section to this orbit so that these two groups of functions provide the splitting of $\mathcal{A}_\lambda$ in the sense of the Weinstein theorem,
- the Casimir functions of $\mathcal{A}_\lambda$ are $z_1, \dots, z_{l-1}$ and $f(u_1, u_2, u_3)$, where f is a non-degenerate quadratic form in u_1, u_2, u_3, which can be naturally identified with the quadratic Casimir of the Lie algebra $\mathfrak{u}$,
- $p_1, \dots, p_s$ belong to $\mathcal{F}_\mathcal{P}$,
- if $f \in \mathcal{F}_\mathcal{P}$, then $\frac{\partial f}{\partial q_i} \equiv 0$, i.e., $f = f(z, u, p)$,
- there exists a function $g \in \mathcal{F}_\mathcal{P}$ such that $\big(\frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}, \frac{\partial g}{\partial u_3}\big) \neq 0$.

It follows immediately from this description that the Lagrangian fibration associated with the commuting family $\mathcal{F}_\mathcal{P}$ is locally given by very simple functions $z_1, \dots, z_{l-1}$, $p_1, \dots, p_s$, $f(u_1, u_2, u_3)$, and $g(z, u, p)$. Hence, the structure of the singularity at x is completely determined by the quadratic form $f(u_1, u_2, u_3)$ and 3-vector $\big(\frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}, \frac{\partial g}{\partial u_3}\big) \neq 0$.

We now give a more algebraic interpretation of this construction and a sufficient non-degeneracy condition for $x \in M$.

Consider another bracket $\mathcal{A}_\mu$ from $\mathcal{P}$, $\mu \neq \lambda$, and take the restriction of $\mathcal{A}_\mu(x)$ to $\operatorname{Ker} \mathcal{A}_\lambda(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$. It follows easily from the Jordan–Kronecker theorem that $\operatorname{Ker}\big(\mathcal{A}_\mu(x)|_{\operatorname{Ker} \mathcal{A}_\lambda(x)}\big)$ has codimension 2 in $\operatorname{Ker} \mathcal{A}_\lambda(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$. It can be easily checked that the center $\mathbb{R}^{l-1}$ belongs to $\operatorname{Ker}\big(\mathcal{A}_\mu(x)|_{\operatorname{Ker} \mathcal{A}_\lambda(x)}\big)$. This means that the restriction of $\mathcal{A}_\mu$ to $\mathfrak{u}$ has rank 2 and $\operatorname{Ker}(\mathcal{A}_\mu(x)|_\mathfrak{u})$ is generated by some vector $\xi \in \mathfrak{u}$. It turns out that non-degeneracy condition can be naturally formulated in terms of this vector ξ (in fact, this ξ is an analog of the 3-vector $\big(\frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}, \frac{\partial g}{\partial u_3}\big) \neq 0$ mentioned above).

Theorem 5. *Let x be a corank 1 critical point of $\mathcal{F}_\mathcal{P}$. Suppose that*

1) *there exists unique $\lambda \in \overline{\mathbb{R}}$ such that $\operatorname{rank} \mathcal{A}_\lambda(x) < \operatorname{rank} \mathcal{P}$,*
2) *the bracket $\mathcal{A}_\lambda$ is semisimple, i.e., $(M, \mathcal{A}_\lambda)$ has a natural identification with the dual space $\mathfrak{g}^*$ of a real semisimple Lie algebra $\mathfrak{g}$ endowed with the standard Lie–Poisson bracket,*
3) *x is a semisimple singular element in $\mathfrak{g}^* = \mathfrak{g}$, and $\operatorname{Ker} \mathcal{A}_\lambda(x) = \mathfrak{u} \oplus \mathbb{R}^{l-1}$, where $\mathfrak{u}$ is a three-dimensional semisimple subalgebra, $l = \operatorname{ind} \mathfrak{g}$,*
4) *$\operatorname{Ker}(\mathcal{A}_\mu(x)|_\mathfrak{u})$ is generated by $\xi \in \mathfrak{u}$, $\xi \neq 0$, $\mu \neq \lambda$.*

If ξ is semisimple element in $\mathfrak{u}$, then x is non-degenerate. Moreover, if $\langle \xi, \xi \rangle > 0$, then the singularity is hyperbolic, and if $\langle \xi, \xi \rangle < 0$, then the singularity is elliptic, where $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{u}$.

To prove this statement we use the following modification of Definition 2.

Let $\mathcal{F}$ be a complete commutative family of functions on a Poisson manifold $(M, \mathcal{A})$. Let x be a regular point in M in the sense that $x \notin S_\mathcal{A}$. Suppose that x is a corank 1 singularity, i.e., $\dim d\mathcal{F}(x) = \frac{1}{2}(\dim M + \operatorname{corank} \mathcal{A}) - 1$. Assume that there exists a function $f \in \mathcal{F}$ such that $df(x) = 0$. Consider the linearization of the Hamiltonian vector field $\operatorname{sgrad}_\mathcal{A} f$ at the point x, i.e., the linear operator defined as $\mathcal{A} \cdot d^2 f(x)$. If this linearization has a non-zero eigenvalue, than x is non-degenerate.

In our case the commutative family $\mathcal{F}_\mathcal{P}$ possesses a very nice function f which can be considered as the quadratic Casimir for the subalgebra $\mathfrak{u}$.

This function can be described explicitly as follows. Each semisimple Lie algebra $\mathfrak{g}$ possesses one remarkable Casimir function f, which is the product of all the roots (both positive and negative). It is easy to see that this function vanishes on the set Sing of singular elements of $\mathfrak{g}$. Moreover, its

differential vanishes on Sing as well. We shall use f to verify the non-degeneracy of x by computing the corresponding linearized Hamiltonian vector field $\mathcal{A}_\mu \cdot d^2 f(x)$.

It is easy to see that under the assumptions of Theorem 5, the set of singular points Sing in a neighborhood of x is a smooth submanifold in $\mathfrak{g}$ of codimension 3, and the Lie algebra $\mathfrak{g}$ can be presented as the direct (orthogonal) sum of the 3-dimensional semisimple subalgebra $\mathfrak{u}$ and the tangent space $T_x \operatorname{Sing}$. The Hessian of f at $x \in \operatorname{Sing}$ is very simple: its kernel is $T_x \operatorname{Sing}$ and the restriction of $d^2 f(x)$ to $\mathfrak{u}$ is the Killing form on $\mathfrak{u}$ (up to a non-zero factor).

The question we are interested in is about the non-zero eigenvalues of the operator $\mathcal{A}_\mu \cdot d^2 f(x)$. Since $T_x \operatorname{Sing}$ is the kernel of $d^2 f(x)$, this question can be naturally reformulated for the restriction of the form $\mathcal{A}_\mu(x)$ to $\mathfrak{u}$. Namely, we now have the following situation: $\mathfrak{u}$ is a three-dimensional Lie algebra, A is a non-trivial skew-symmetric bilinear form on $\mathfrak{u}$ and f is the quadratic Casimir function on $\mathfrak{u}^*$ which coincides with the Killing form $\langle\ ,\ \rangle$ on $\mathfrak{u}$ under the standard identification $\mathfrak{u}^* \sim \mathfrak{u}$. We wonder if the operator $A \cdot d^2 f$ has a non-zero eigenvalue. The answer is given by the following simple

Proposition 8. *Let $\xi \in \mathfrak{u}$ be a vector that generates the kernel of A. Then*

1) *$A \cdot d^2 f$ has two non-zero eigenvalues if and only if ξ is semisimple (equivalently, $\langle \xi, \xi \rangle \neq 0$),*
2) *the eigenvalues of $A \cdot d^2 f$ are pure imaginary $i\alpha, -i\alpha$, $\alpha \neq 0$, if and only if $\langle \xi, \xi \rangle < 0$,*
3) *the eigenvalues of $A \cdot d^2 f$ are real $\alpha, -\alpha$, $\alpha \neq 0$, if and only if $\langle \xi, \xi \rangle > 0$.*

This statement immediately implies Theorem 5.

Notice that the real three-dimensional Lie algebra $\mathfrak{u}$ can be of two different types: either $\mathrm{so}(3)$ or $\mathrm{sl}(2, \mathbb{R})$. In the first case, all non-zero vectors $\xi \in \mathrm{so}(3)$ are semisimple and $\langle \xi, \xi \rangle < 0$. Thus, in this situation a critical point x of corank 1 is automatically non-degenerate and of elliptic type. For $\mathfrak{u} = \mathrm{sl}(2, \mathbb{R})$ all situations are possible: x can be either elliptic, or hyperbolic, or degenerate.

We now apply the sufficient non-degeneracy condition given by Theorem 5 to our two model examples.

Example A (Corank 1 Singularities). Let $x \in \mathfrak{g}$ be a corank 1 critical point for the momentum mapping $\Phi_a : \mathfrak{g} \to \mathbb{R}^s$. Then, as we know from the general construction discussed above, there exists unique $\lambda \in \mathbb{R}$ such that $x + \lambda a$ is a singular element in $\mathfrak{g}$.

Assume that x is semisimple, take its centralizer $\mathfrak{g}^{x+\lambda a} = \{\xi \in \mathfrak{g} \mid [\xi, x + \lambda a] = 0\}$, and extract its semisimple part $\mathfrak{u} \subset \mathfrak{g}^{x+\lambda a}$. We are interested in the restriction of the bracket $\{\ ,\ \}_a$ to $\mathfrak{u}$:

$$A(\xi_1, \xi_2) = \langle a, [\xi_1, \xi_2] \rangle, \qquad \xi_1, \xi_2 \in \mathfrak{u}.$$

In this expression we can replace a by its orthogonal projection $b = \operatorname{pr}_{\mathfrak{u}} a \neq 0$ onto $\mathfrak{u}$:

$$A(\xi_1, \xi_2) = \langle a, [\xi_1, \xi_2] \rangle = \langle b, [\xi_1, \xi_2] \rangle$$

It is easy to see that the kernel of this form is generated by $b \in \mathfrak{u}$. Thus, the general construction implies the following

Theorem 5. *Let $x \in \mathfrak{g}$ be a critical point of corank 1, and $\lambda \in \mathbb{R}$ the unique value of the parameter such that $x + \lambda a$ is a singular element in $\mathfrak{g}$. Assume that $x + \lambda a$ is semisimple and $\mathfrak{u}$ is the semisimple part of the centralizer $\mathfrak{g}^{x+\lambda a}$. Consider the natural orthogonal projection $b = \operatorname{pr}_{\mathfrak{u}} a$ of a onto $\mathfrak{u}$. If b is semisimple and non-zero, then x is non-degenerate. Moreover, if $\langle b, b \rangle > 0$, then the singularity is hyperbolic, and if $\langle b, b \rangle < 0$, then the singularity is elliptic, where $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{u}$.*

In particular, in the case of a compact Lie algebra $\mathfrak{g}$, all corank 1 singularities are non-degenerate and of elliptic type.

Example B (Corank 1 Singularities). For the Manakov case, the situation is a bit more complicated. Let $X \in \mathrm{so}(n)$ be a corank 1 critical point. For definiteness we assume that the standard bracket $\{\ ,\ \}_E$ has maximal rank at X (in other words, X is a regular element in $\mathrm{so}(n)$).

Among all values of the parameter, there is exactly one $\lambda \in \mathbb{R}$ such that the rank of the bracket $\{\ ,\ \}_{C+\lambda E}$ at the point X drops by 2. If $\det(C + \lambda E) \neq 0$, then the corresponding Lie

algebra $\mathfrak{g}_\lambda$ (which is the space of skew-symmetric matrices with the commutator $[\xi_1,\xi_2]_{C_\lambda} = \xi_1 C_\lambda \xi_2 - \xi_2 C_\lambda \xi_1$, where $C_\lambda = C + \lambda E$) is naturally isomorphic to the Lie algebra of the group of linear transformations preserving the form C_λ. By definition, the latter Lie algebra, which we denote by $\mathrm{so}(C_\lambda)$, consists of matrices η satisfying the equation $\eta^\top C_\lambda + C_\lambda \eta = 0$.

Consider the mapping

$$\varphi : \mathfrak{g}_\lambda \to \mathrm{so}(C_\lambda), \qquad \varphi(\xi) = \xi C_\lambda = \eta.$$

It is easy to see that φ establishes an isomorphism between $\mathfrak{g}_\lambda$ and $\mathrm{so}(C_\lambda)$:

$$\varphi[\xi_1,\xi_2]_{C_\lambda} = (\xi_1 C_\lambda \xi_2 - \xi_2 C_\lambda \xi_1) C_\lambda = [\xi_1 C_\lambda, \xi_2 C_\lambda] = [\varphi(\xi_1), \varphi(\xi_2)].$$

Consider another mapping between the same spaces:

$$\psi : \mathfrak{g}_\lambda \to \mathrm{so}(C_\lambda), \qquad \psi(X) = C_\lambda^{-1} X = Y.$$

This mapping gives a natural identification between the dual spaces $\mathfrak{g}_\lambda^*$ and $\mathrm{so}(C_\lambda)^*$ and can be treated, in fact, as $(\varphi^*)^{-1}$.

The kernel of $\{\ ,\ \}_{C_\lambda}$ at the point X is given by the equation $C_\lambda \xi X - X \xi C_\lambda = 0$. Using the above transformations $X = C_\lambda Y$, $\xi = \eta C_\lambda^{-1}$, we rewrite it as

$$\eta Y - Y \eta = 0.$$

Thus, η belongs to the centralizer of Y in the Lie algebra $\mathrm{so}(C_\lambda)$ (as expected, of course!). If we assume that Y is a semisimple element in $\mathrm{so}(C_\lambda)$, then its centralizer contains a well-defined semisimple subalgebra $\mathfrak{u}$ of dimension 3. We now need to restrict the standard 2-form related to the bracket $\{\ ,\ \}_E$ onto $\mathfrak{u}$ and find the kernel of this restriction.

In the original notation, this restriction can be written as

$$A(\xi_1,\xi_2) = \mathrm{Tr}(X[\xi_1,\xi_2]) = \mathrm{Tr}(\xi_1\xi_2 X - \xi_2\xi_1 X), \qquad \xi_i \in \varphi^{-1}(\mathfrak{u}).$$

Passing to the Lie algebra $\mathrm{so}(C_\lambda)$ (i.e., from ξ and X to η and Y), we obtain

$$A(\xi_1,\xi_2) = \mathrm{Tr}(\eta_1 C_\lambda^{-1}\eta_2 Y - \eta_2 C_\lambda^{-1}\eta_1 Y) = \mathrm{Tr}(\eta_1 C_\lambda^{-1} Y \eta_2 - \eta_2 C_\lambda^{-1} Y \eta_1) = \mathrm{Tr}((\eta_2\eta_1 - \eta_1\eta_2) C_\lambda^{-1} Y).$$

Let $\pi(C_\lambda^{-1}Y)$ denotes the orthogonal projection of the matrix $C_\lambda^{-1}Y$ onto the Lie subalgebra $\mathfrak{u}$ (with respect to the standard form $\langle A, B\rangle = \mathrm{Tr}\, AB$ defined on $\mathrm{gl}(n,\mathbb{R})$). It is clear that

$$A(\xi_1,\xi_2) = \mathrm{Tr}((\eta_2\eta_1 - \eta_1\eta_2)\pi(C_\lambda^{-1}Y)) = \mathrm{Tr}(\eta_2[\eta_1, \pi(C_\lambda^{-1}Y)]).$$

Thus, the kernel of $\{\ ,\ \}_E$ restricted to $\mathfrak{u}$ is simply the centralizer of the $\pi(C_\lambda^{-1}Y)$ in $\mathfrak{u}$. Since X is a corank 1 singularity, this kernel is one-dimensional and therefore is generated by the element $\pi(C_\lambda^{-1}Y) \neq 0$ itself. Thus, we come to the following conclusion.

Theorem 5. *Let X be a corank 1 critical point for $\mathcal{F}_C$. Then there exists a unique value $\lambda \in \mathbb{R}$ such that $X \in S_\lambda$. Assume that the following conditions are fulfilled*:

1) $\det C_\lambda \neq 0$ *so that $\mathfrak{g}_\lambda$ is semisimple,*
2) $Y = C_\lambda^{-1}X$ *is semisimple in* $\mathrm{so}(C_\lambda)$ (*this element is automatically singular*),
3) *the orthogonal projection of $C_\lambda^{-1}Y = C_\lambda^{-2}X$ onto $\mathfrak{u}$ is semisimple, where $\mathfrak{u}$ is the semisimple* (3*-dimensional*) *part of the centralizer of $Y = C_\lambda^{-1}X$ in* $\mathrm{so}(C_\lambda)$.

Then X is non-degenerate. Moreover, if $\mathrm{Tr}(\pi(C_\lambda^{-1}Y))^2 > 0$, then X is hyperbolic, and if $\mathrm{Tr}(\pi(C_\lambda^{-1}Y))^2 < 0$, then X is elliptic.

ACKNOWLEDGMENTS

This work was supported by RFBR (grant 07-01-00648), the President program of Support for Leading Scientific Schools of Russia (grant NSh-660.2008.1) and the Federal target program "Scientific and scientific-pedagogical personnel of innovative Russia" (project 2009-1.5-503-004-019).

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