Complete involutive algebras of functions on cotangent bundles of homogeneous spaces

Alexey V. Bolsinov
Department of Mechanics and Mathematics,
119899, Moscow, Russia
e-mail: bolsinov@mech.math.msu.su

Božidar Jovanović
Matematički Institut SANU, Kneza Mihaila 35,
11000 Beograd, Serbia, Yugoslavia
e-mail: bozaj@mi.sanu.ac.yu

Abstract

Homogeneous spaces of all compact Lie groups admit Riemannian metrics with completely integrable geodesic flows by means of C^∞–smooth integrals [9, 10]. The purpose of this paper is to give some constructions of complete involutive algebras of analytic functions, polynomial in velocities, on the (co)tangent bundles of homogeneous spaces of compact Lie groups. This allows us to obtain new integrable Riemannian and sub-Riemannian geodesic flows on various homogeneous spaces, such as Stiefel manifolds, flag manifolds and orbits of the adjoint actions of compact Lie groups.

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Contents

0 Introduction

Let M be a $2n$–dimensional symplectic manifold with the canonical Poisson bracket $\{\cdot,\cdot\}_M$. The *complete integrability* of the Hamiltonian equations with a Hamiltonian function H

$$\dot{x}_i = \{x_i, H\}_M, \quad i = 1, \dots 2n, \tag{1}$$

means that there are n smooth involutive integrals $f_1, \dots, f_n$ whose differentials are independent on an open dense set of M. The algebra of integrals $\mathcal{F} = \{f_1, \dots, f_n\}$ is usually called *complete involutive (or commutative) algebra*.

Let $M = T^*Q$ be the cotangent bundle of a Riemannian manifold (Q, g) with the natural symplectic structure. It is well known that the geodesic flow on Q is described by the Hamiltonian equations (1) with the Hamiltonian $H(p,q) = \frac{1}{2}g_q^{-1}(p,p)$, $p \in T_q^*Q$. The geodesic flow of a generic metric is non-integrable [2]. Moreover there are very serious topological obstructions to the integrability [2, 31].

Let us briefly recall known results concerning the integrability of geodesic flows on compact homogeneous spaces. Mishchenko and Fomenko obtained integrability of certain left-invariant metrics on compact Lie groups [20, 35]. This result was generalized to symmetric spaces by Thimm [33], Mishchenko [19], Brailov [12] and Mikityuk [22]. Integrable geodesic flows on the sphere S^2 and torus $\mathbb{T}^2$ with an additional integral polynomial in momenta were studied in [18, 3, 6, 28, 29, 16]. Homogeneous but not symmetric spaces, with integrable geodesic flows can be found in [33, 13, 27, 7, 8, 24]. Examples given in [13, 7, 8] are homogeneous spaces G/Γ of non-compact groups where $\Gamma \subset G$ is a discrete cocompact subgroup. The corresponding geodesic flows are integrable by smooth integrals, and by Taimanov's theorem [31] can not be integrable by analytic ones.

Recently, we have proved the *integrability in the non-commutative sense* of the geodesic flows of bi-invariant metrics ds_0^2 on all homogeneous spaces $\mathfrak{G}/\mathfrak{H}$ where $\mathfrak{G}$ is a compact Lie group [9]. Their first integrals are analytic functions polynomial in velocities. Also, we have shown that the non-commutative integrability implies the classical commutative integrability by means of C^∞–smooth integrals [10].

In this paper we are going to consider a more delicate problem: the construction of complete commutative algebras of integrals of ds_0^2 that are *polynomial in velocities.* Let $G = H + V$ be the orthogonal decomposition of the Lie algebra of the group $\mathfrak{G}$ ($G = T_e\mathfrak{G}$, $H = T_e\mathfrak{H}$). Then the problem is reduced to a problem of finding complete involutive algebras of $\mathrm{Ad}_{\mathfrak{H}}$ invariant polynomials on V (section 1). Two natural methods for construction of commutative families of $\mathrm{Ad}_{\mathfrak{H}}$ invariant functions: shift-argument method (sections 3) and chains of subalgebras method (sections 4, 5) are presented. As a main result we have proved that in many examples (Stiefel manifolds, flag manifolds, orbits of the adjoint actions of compact Lie groups etc.) those methods lead to complete commutative algebras. In the last section we use these algebras to construct $\mathfrak{G}$–invariant integrable sub-Riemannian geodesic flows on various homogeneous spaces $\mathfrak{G}/\mathfrak{H}$.

1 Integrable pairs

1.1 Non-commutative integrability

We shall briefly recall the concept of non-commutative integrability introduced by Mishchenko and Fomenko in [21]. Let $(M, \{\cdot,\cdot\}_M)$ be a Poisson manifold and $\mathcal{F}$ be a Poisson subalgebra in $C^\infty(M)$. Let F_x be the subspace of $T^*_x M$ generated by $df(x)$, $f \in \mathcal{F}$. Suppose that $\dim F_x = l$ and $\dim \ker\{\cdot,\cdot\}_M(x)|_{F_x} = d$ almost everywhere. The numbers l and d are usually denoted by $\operatorname{ddim} \mathcal{F}$ and $\operatorname{dind} \mathcal{F}$. The algebra $\mathcal{F}$ is called *complete* if

$$\operatorname{ddim} \mathcal{F} + \operatorname{dind} \mathcal{F} = \dim M + \operatorname{corank} \{\cdot,\cdot\}_M. \tag{2}$$

(for symplectic manifolds $\operatorname{corank} \{\cdot,\cdot\}_M = 0$). In particular, we shall say that $\mathcal{F}$ is *complete at* $x \in M$ if $\dim F_x + \dim \ker\{\cdot,\cdot\}_M(x)|_{F_x} = \dim M + \operatorname{corank} \{\cdot,\cdot\}_M(x)|_{F_x}$. *A complete subalgebra* $\mathcal{F}'$ of $\mathcal{F}$ is defined by $\operatorname{ddim} \mathcal{F}' + \operatorname{dind} \mathcal{F}' = \operatorname{ddim} \mathcal{F} + \operatorname{dind} \mathcal{F}$.

The Hamiltonian system (1) on a symplectic manifold M is *completely integrable in the non-commutative sense* if it possesses a complete algebra of first integrals. Then each connected compact component of a regular level-set of functions $f_1, \dots, f_l \in \mathcal{F}$ is a d–dimensional invariant isotropic torus $\mathbb{T}^d$. Moreover, in a neighborhood of $\mathbb{T}^d$ there are *generalized action-angle variables* p, q, I, φ such that the symplectic form becomes $\omega = \sum_{i=1}^d dI_i \wedge d\varphi_i + \sum_{i=1}^k dp_i \wedge dq_i$, and the Hamiltonian function H depends only on $I_1, \dots, I_d$. Thus the motion on the invariant tori is quasi-periodic and given by:

$$\dot\varphi_1 = \omega_1(I) = \frac{\partial H}{\partial I_1}, \dots, \dot\varphi_d = \omega_d(I) = \frac{\partial H}{\partial I_d}, \quad \dot I = \dot p = \dot q = 0.$$

If the algebra of integrals $\mathcal{F} = \{f_1, \dots, f_n\}$ is commutative ($\operatorname{ddim} \mathcal{F} = \operatorname{dind} \mathcal{F} = n$) then we have the usual Liouville integrability.

Different versions of this result can be found in [2, 35].

1.2 Algebras of functions on $T(\mathfrak{G}/\mathfrak{H})$

Let $\mathfrak{G}$ be a connected compact Lie group, $\mathfrak{H}$ a connected Lie subgroup, $Q = \mathfrak{G}/\mathfrak{H}$ the corresponding homogeneous space, $G = T_e\mathfrak{G}$, $H = T_e\mathfrak{H}$ the Lie algebras of $\mathfrak{G}$ and $\mathfrak{H}$. Let $G = H + V$ be the orthogonal decomposition of G according a non-degenerate $\mathrm{Ad}_{\mathfrak{G}}$–invariant scalar product $\langle\cdot,\cdot\rangle$. We can identify V with the tangent space $T_{\pi(e)}Q$, where $\pi : \mathfrak{G} \to Q$ is the canonical projection. The scalar product induces the $\mathfrak{G}$–invariant metric ds_0^2 on Q. We shall use this metric and the scalar product to identify T^*Q with TQ and G^* with G.

Let $\Phi : TQ \to G$ be the moment map of the $\mathfrak{G}$ action on TQ (it has the form $\Phi(gv) = \mathrm{Ad}_g v$, where by gv we denote the action of $g \in \mathfrak{G}$ on an element $v \in V = T_{\pi(e)}Q$).

The two following natural classes of functions on TQ are integrals of the geodesic flow of ds_0^2: $\mathcal{F}_1 = \{h \circ \Phi,\ h : G \to \mathbb{R}\}$, induced by the moment map (Noether integrals), and $\mathcal{F}_2$, $\mathfrak{G}$–invariant functions on TQ [9]. Note that the functions from these classes commute: $\{\mathcal{F}_1, \mathcal{F}_2\}_{TQ} = 0$.

$\mathfrak{G}$–invariant functions on TQ are in one-one correspondence with $\mathrm{Ad}_{\mathfrak{H}}$ invariant functions on V. Recall that a function $p : V \to \mathbb{R}$ is an invariant of the adjoint action of $\mathfrak{H}$ on V if and only if it satisfies the following equation:

$$\langle \mathrm{ad}_H v, \nabla p(v) \rangle = 0, \quad v \in V$$

or $pr_H[\nabla p(v), v] = 0$, where pr_H denotes the orthogonal projection of G to H.

Let τ_1 be the mapping which assigns to every function h on G, the function $h \circ \Phi$ and let τ_2 be the mapping which realizes the correspondence between $\mathrm{Ad}_{\mathfrak{H}}$–invariant functions on V and $\mathfrak{G}$–invariant functions on TQ.

Denote by $\mathbb{R}[G]$ the algebra of polynomials on G and by $\mathbb{R}[V]^{\mathfrak{H}}$ the algebra of $\mathrm{Ad}_{\mathfrak{H}}$–invariant polynomials on V. In such a notation we have that $\mathcal{F}_1 = \tau_1(\mathbb{R}[G])$ and $\mathcal{F}_2 = \tau_2(\mathbb{R}[V]^{\mathfrak{H}})$.

Let $\{\cdot,\cdot\}_G$ be the Lie-Poisson bracket on G:

$$\{f(x), g(x)\}_G = \langle x, [\nabla f(x), \nabla g(x)] \rangle, \ f, g : G \to \mathbb{R},$$

and let

$$\{f(v), g(v)\}_V = \langle v, [\nabla f(v), \nabla g(v)] \rangle, \ f, g : V \to \mathbb{R}.$$

To simplify notation, throughout the paper we shall use the same symbol for the gradients of functions on G and V. For example, if f is a function on G and $f|_V$ its restriction to V then $\nabla f|_V(v) = pr_V(\nabla f(v))$.

Notice that $\{\cdot,\cdot\}_V$ is a Poisson bracket on $\mathbb{R}[V]^{\mathfrak{H}}$, i.e., satisfies the Jacobi identity. However this condition fails if instead of $\mathbb{R}[V]^{\mathfrak{H}}$ we consider the algebra $\mathbb{R}[V]$ of all polynomials on V.

We have that

$$\begin{aligned} \mathrm{ddim}\ \mathbb{R}[V]^{\mathfrak{H}} &= \dim V - \dim H + \dim \mathrm{ann}_H(v), \\ \mathrm{dind}\ \mathbb{R}[V]^{\mathfrak{H}} &= \dim pr_V(\mathrm{ann}_G(v)), \end{aligned} \tag{3}$$

for generic $v \in V$ [9]. Here $\mathrm{ann}_G(v)$ and $\mathrm{ann}_H(v)$ denote the annihilators of v in G and H respectively:

$$\mathrm{ann}_G(v) = \{\eta \in G, [\eta, v] = 0\}, \quad \mathrm{ann}_H(v) = \{\eta \in H, [\eta, v] = 0\}.$$

By genericity of $v \in V$ we mean that the dimensions of $\mathrm{ann}_G(v)$ and $\mathrm{ann}_H(v)$ are minimal.

Remark 1.1 The kernel of the Poisson structure $\{\cdot,\cdot\}_V$ on $\mathbb{R}[V]^{\mathfrak{H}}$ at a generic point $v \in V$ is spanned by the gradients of the polynomials that can be obtain as restrictions to V of the functions from $\mathcal{F}_1 \cap \mathcal{F}_2$ (these are central functions of $\mathbb{R}[V]^{\mathfrak{H}}$). On the other hand, the functions in $\mathcal{F}_1 \cap \mathcal{F}_2$ are of the form $h \circ \Phi$, where h is an invariant of the adjoint representation. Moreover dind $(\mathcal{F}_1 + \mathcal{F}_2) =$ dind $\mathcal{F}_1 =$ dind $\mathcal{F}_2 =$ ddim $(\mathcal{F}_1 \cap \mathcal{F}_2) =$ dind $\mathbb{R}[V]^{\mathfrak{H}}$.

Remark 1.2 The number $\frac{1}{2}(\operatorname{ddim}\mathbb{R}[V]^{\mathfrak{H}} - \operatorname{dind}\mathbb{R}[V]^{\mathfrak{H}})$ is equal to the codimension $\delta(\mathfrak{G}^{\mathbb{C}}, \mathfrak{H}^{\mathbb{C}})$ of maximal dimension orbits of the Borel subgroup $\mathfrak{B} \subset \mathfrak{G}^{\mathbb{C}}$ in the complex algebraic variety $\mathfrak{G}^{\mathbb{C}}/\mathfrak{H}^{\mathbb{C}}$ (see [25]). The number $\delta(\mathfrak{G}^{\mathbb{C}}, \mathfrak{H}^{\mathbb{C}})$ is called the *complexity* of $\mathfrak{G}^{\mathbb{C}}/\mathfrak{H}^{\mathbb{C}}$ [26].

The mappings τ_1 and τ_2 are a homomorphism and an anti-homomorphism of Poisson structures [33]:

$$\{h_1, h_2\}_G(\operatorname{Ad}_g v) = \{\tau_1(h_1), \tau_1(h_2)\}_{TQ}(gv),$$
$$\{p_1, p_2\}_V(v) = -\{\tau_2(p_1), \tau_2(p_2)\}_{TQ}(gv).$$

Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are commutative subalgebras in $\mathbb{R}[G]$ and $\mathbb{R}[V]^{\mathfrak{H}}$ respectively, then $\tau_1(\mathcal{A}) \subset \mathcal{F}_1$ and $\tau_2(\mathcal{B}) \subset \mathcal{F}_2$ will be commutative subalgebras of $\mathcal{F}_1$ and $\mathcal{F}_2$.

1.3 Integrable geodesic flows

Theorem 1.1 *[9] (i) The geodesic flow of ds_0^2 on $Q = \mathfrak{G}/\mathfrak{H}$ is completely integrable in the non-commutative sense. The complete algebra of integrals is $\mathcal{F}_1 + \mathcal{F}_2$.*

(ii) Suppose that the following conditions hold:

(A) $\mathcal{A} \subset \mathbb{R}[G]$ is a complete commutative algebra on the Ad*–orbits $\mathcal{O}_{\mathfrak{G}}(v)$, for generic $v \in V$;*

(B) $\mathcal{B}$ is a complete commutative subalgebra of $\mathbb{R}[V]^{\mathfrak{H}}$.

Then $\tau_1(\mathcal{A}) + \tau_2(\mathcal{B})$ is a complete commutative algebra on TQ.

Remark 1.3 From (3), the condition (B) can be rewritten in the following form:

$$\operatorname{ddim}\mathcal{B} = \frac{1}{2}(\operatorname{ddim}\mathbb{R}[V]^{\mathfrak{H}} + \operatorname{dind}\mathbb{R}[V]^{\mathfrak{H}}) = \dim V - \frac{1}{2}\dim\mathcal{O}_{\mathfrak{G}}(v), \tag{4}$$

for generic $v \in V$. If a generic element $v \in V$ is regular in G, then (B) becomes:

$$\operatorname{ddim}\mathcal{B} = \dim V - \frac{1}{2}(\dim G - \operatorname{rank} G) = \frac{1}{2}(\dim G + \operatorname{rank} G) - \dim H, \tag{5}$$

Recall that $v \in V$ is *regular* in G if $\operatorname{ann}_G(v)$ is a commutative algebra, or equivalently if $\dim\operatorname{ann}_G(v) = \operatorname{rank} G$.

Let $\tau_1(\mathcal{A}) + \tau_2(\mathcal{B})$ be a complete commutative subalgebra of $\mathcal{F}_1 + \mathcal{F}_2$. Let $h_1 \in \mathcal{A}$ and $h_2 \in \mathcal{B}$ be quadratic positive definite polynomials. Let ds_1^2, ds_2^2 be the Riemannian metrics on $\mathfrak{G}/\mathfrak{H}$ that correspond to the Hamiltonians $H_1 = \tau(h_1)$ and $H_2 = \tau(h_2)$ respectively. Then the geodesic flows of metrics $\lambda_1 ds_1^2 + \lambda_2 ds_2^2$ will be completely integrable and the dimension of invariant tori, in general, will be equal to $\dim\mathfrak{G}/\mathfrak{H}$. The geometrical meaning of metrics ds_1^2 and ds_2^2 can be found in [9]. Also, we can take ds_2^2 to be a $\mathfrak{G}$–invariant sub-Riemannian metric on $\mathfrak{G}/\mathfrak{H}$. In such a way we can get integrable sub-Riemannian geodesic flows (see section 6).

There is a well known construction, called the argument shift method [20], which allows us to obtain a complete commutative family of polynomials on every Ad–orbit of a compact Lie group [20, 5, 12, 22]. Thus, to construct a complete commutative algebra of functions on TQ we need to find a complete commutative subalgebra $\mathcal{B}$ in $\mathbb{R}[V]^{\mathfrak{H}}$. This leads us to the following definition.

Definition 1.1 We will call $(\mathfrak{G}, \mathfrak{H})$ an integrable pair, if there exists a complete commutative subalgebra $\mathcal{B}$ in $\mathbb{R}[V]^{\mathfrak{H}}$.

Besides constructing metrics with integrable geodesic flows on homogeneous spaces $\mathfrak{G}/\mathfrak{H}$, the question whether a pair $(\mathfrak{G},\mathfrak{H})$ is integrable or not is related to the Mischenko–Fomenko conjecture saying that all non-commutatively integrable systems are integrable in the usual commutative sense by means of integrals from the *same functional class* as original non-commuting integrals [21, 35]. We have proved the conjecture in the smooth case [10]. For the non-commutatively integrable geodesic flow of the metric ds_0^2 on $\mathfrak{G}/\mathfrak{H}$, the Mischenko–Fomenko conjecture can be stated as follows.

Conjecture 1.1 All pairs $(\mathfrak{G},\mathfrak{H})$ are integrable.

We can summarize the above considerations as follows.

Theorem 1.2 *If $(\mathfrak{G},\mathfrak{H})$ is an integrable pair then the geodesic flow of ds_0^2 is completely integrable in the commutative sense by means of analytic integrals, polynomial in velocities.*

2 Simplest examples

2.1 Spherical pairs

Consider a homogeneous space $Q=\mathfrak{G}/\mathfrak{H}$ of a compact Lie group $\mathfrak{G}$. If the algebra $\mathcal{F}_2$ of $\mathfrak{G}$–invariant functions on TQ is commutative (i.e., $\operatorname{ddim}\mathbb{R}[V]^{\mathfrak{H}}=\operatorname{dind}\mathbb{R}[V]^{\mathfrak{H}}$) then $(\mathfrak{G},\mathfrak{H})$ is called a *Gelfand* or *spherical pair* (see [36]). By definition, spherical pairs are integrable.

In this case, in a neighborhood of a generic point $x\in TQ$ each $\mathfrak{G}$–invariant function can be expressed as a function of Noether integrals. In other words:

$$\operatorname{span}\{df(x),\ f\in\mathcal{F}_2\}\subset\operatorname{span}\{df(x),\ f\in\mathcal{F}_1\}\subset T_x^*(T(\mathfrak{G}/\mathfrak{H})).$$

So we can use just Noether integrals to get integrability of any $\mathfrak{G}$–invariant geodesic flow on Q (see [33, 19, 12, 22, 14]).

If $\mathfrak{G}/\mathfrak{H}$ is a symmetric space, then $(\mathfrak{G},\mathfrak{H})$ is a spherical pair, but there exist spherical pairs which are not symmetric. For example, $(SO(n+1)\times SO(n),SO(n))$ is a spherical pair ($SO(n)$ is diagonally embedded in the product).

The spherical pairs have been classified by Kramer (with $\mathfrak{G}$ simple) and Mikityuk and Brion (with $\mathfrak{G}$ semi-simple), see [36] and references therein. Note that for homogeneous spaces $\mathfrak{G}/\mathfrak{H}$ of compact groups $\mathfrak{G}$ the property of $(\mathfrak{G},\mathfrak{H})$ to be a spherical pair is equivalent to the property that $\mathfrak{G}/\mathfrak{H}$ is *weakly symmetric* (see [36]).

2.2 Almost spherical pairs

Following [24, 25], we shall say that $(\mathfrak{G},\mathfrak{H})$ is an *almost spherical pair* if the algebra $\mathbb{R}[V]^{\mathfrak{H}}$ of $\mathrm{Ad}_{\mathfrak{H}}$–invariant polynomials on V satisfies:

$$\operatorname{ddim}\mathbb{R}[V]^{\mathfrak{H}}=2+\operatorname{dind}\mathbb{R}[V]^{\mathfrak{H}},\tag{6}$$

or equivalently, if the complexity of $\mathfrak{G}^{\mathbb{C}}/\mathfrak{H}^{\mathbb{C}}$ is equal to one (remark 1.2). Using (3) we can rewrite (6) as follows:

$$\dim V-\dim H+2\dim\operatorname{ann}_H(v)-\dim\operatorname{ann}_G(v)-2=0,\tag{7}$$

for a generic element $v\in V$.

Almost spherical pairs are integrable. As a complete commutative subalgebra $\mathcal{B} \subset \mathbb{R}[V]^{\mathfrak{H}}$ we can take the algebra generated by the central functions of $\mathbb{R}[V]^{\mathfrak{H}}$ and one arbitrary non-central function in $\mathbb{R}[V]^{\mathfrak{H}}$.

Panyushev [26] and Mikityuk and Stepin [24] obtained the classification of almost spherical pairs $(\mathfrak{G}, \mathfrak{H})$ with compact connected simple Lie groups $\mathfrak{G}$. Examples of almost spherical pairs are $(SO(n), SO(n-2))$ and $(SU(3), \mathbb{T}^2)$. The integrability of the geodesic flows on $SO(n)/SO(n-2)$ and $SU(3)/\mathbb{T}^2$ was already proved by Thimm [33] and Paternain and Spatzier [27] respectively.

3 Argument shift method

3.1 Argument shift method

Let $\mathbb{R}[G]^{\mathfrak{G}}$ be the algebra of $\mathrm{Ad}_{\mathfrak{G}}$–invariant polynomials on G. Mishchenko and Fomenko showed that the polynomials

$$\mathcal{A}_a = \{p_a^\lambda = p(\cdot + \lambda a),\ \lambda \in \mathbb{R},\ p \in \mathbb{R}[G]^{\mathfrak{G}}\}$$

obtained from the invariants by shifting the argument are all in involution [20]. Furthermore, for *every* adjoint orbit in G, one can find $a \in G$, such that $\mathcal{A}_a$ is a complete involutive set of functions on this orbit. For regular orbits it is proved by Mishchenko and Fomenko [20]. For singular orbits there are several different proofs by Mikityuk [22], Brailov [12] and Bolsinov [5].

Thus, as was already mentioned, the argument shift method allows us to construct a complete commutative subalgebra in $\mathcal{F}_1$. Now we want to use it to construct such a subalgebra in $\mathcal{F}_2$.

By $\mathcal{B}_a$ denote the restriction of $\mathcal{A}_a$ to V:

$$\mathcal{B}_a = \{p(\cdot + \lambda a)|_V,\ \lambda \in \mathbb{R},\ p \in \mathbb{R}[G]^{\mathfrak{G}}\}. \tag{8}$$

Lemma 3.1 *Suppose that*

$$[a, H] = 0, \tag{9}$$

i.e. $H \subset \mathrm{ann}_G(a)$*; then the polynomials from* $\mathcal{B}_a$ *are* $\mathrm{Ad}_{\mathfrak{H}}$*–invariant functions.*

Proof. We shall prove that $p_a^\lambda(x) = p(x + \lambda a)$ is $\mathrm{Ad}_{\mathfrak{H}}$ invariant polynomial on G. Thus, the restriction of p_a^λ to V will be also $\mathrm{Ad}_{\mathfrak{H}}$ invariant.

Since p is $\mathrm{Ad}_{\mathfrak{G}}$–invariant, we have $[\nabla p_a^\lambda(x), x + \lambda a] = 0$. Hence

$$\begin{aligned}\langle \mathrm{ad}_H x, \nabla p_a^\lambda(x)\rangle &= -\langle[\nabla p_a^\lambda(x), x], H\rangle \\ &= -\langle[\nabla p_a^\lambda(x), x + \lambda a], H\rangle + \langle[\nabla p_a^\lambda(x), \lambda a], H\rangle \\ &= -\lambda\langle[H, a], \nabla p_a^\lambda\rangle.\end{aligned} \tag{10}$$

By (9), the last term is equal to zero. Lemma is proved.

Lemma 3.2 *[9] If f_1 and f_2 are in involution $\{f_1, f_2\}_G = 0$ and their restrictions to V: $p_1 = f_1|_V$, $p_2 = f_2|_V$ are $\mathrm{Ad}_{\mathfrak{H}}$–invariant; then $\{p_1, p_2\}_V = 0$.*

From lemmas 3.1, 3.2 and (9) it follows that $\mathcal{B}_a$ is a commutative algebra of $\mathrm{Ad}_{\mathfrak{H}}$–invariant polynomials. The number of independent functions, obtained by shifting the argument, is equal to the dimension of the linear space:

$$B_a = B_a(v) = \mathrm{span}\,\{\nabla p_a^\lambda|_V(v)\} = pr_V \mathrm{span}\,\{\nabla p_a^\lambda(v)\}, \tag{11}$$

for generic $v \in V$. In order to estimate this dimension, we look at the algebra $\mathcal{B}_a$ from the point of view of the bi-Hamiltonian system theory. This approach gives us a possibility to use, in particular, the completeness criterion proved by the first author in [5].

3.2 Compatible Poisson structures

Let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ be compatible Poisson structures on a manifold M. In other words, each linear combination $\lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2$ with constant coefficients is again a Poisson structure. Let

$$\Lambda = \{\lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2,\ \lambda_1,\lambda_2 \in \mathbb{R},\ \lambda_1^2+\lambda_2^2 \neq 0\}.$$

By r denote the rank of a generic bracket in Λ:

$$r = \max_{x\in M, \{\cdot,\cdot\}\in\Lambda} \operatorname{rank} \{\cdot,\cdot\}(x).$$

For each bracket $\{\cdot,\cdot\} \in \Lambda$ of rank r, we consider the set of its Casimir functions. Let $\mathcal{F}_\Lambda$ be the union of these sets. Then $\mathcal{F}_\Lambda$ is involutive set with respect to every Poisson bracket from Λ (see [5]).

Together with Λ, consider its natural complexification $\Lambda^{\mathbb{C}} = \{\lambda_1\{\cdot,\cdot\}_1 + \lambda_2\{\cdot,\cdot\}_2,$ $\lambda_1,\lambda_2 \in \mathbb{C},\ |\lambda_1|^2+|\lambda_2|^2 \neq 0\}$. Here, for $\{\cdot,\cdot\} \in \Lambda^{\mathbb{C}}$, we consider $\{\cdot,\cdot\}(x)$ as a complex valued skew-symmetric bilinear form on the complexification of the co-tangent space $(T_x^*M)^{\mathbb{C}}$.

Let us fix a generic point $x \in M$.

Theorem 3.1 *[5] Let $\{\cdot,\cdot\} \in \Lambda$ and $\operatorname{rank} \{\cdot,\cdot\}(x) = r$. Then $\mathcal{F}_\Lambda$ is complete at $x \in M$ with respect to $\{\cdot,\cdot\}$ if and only if $\operatorname{rank} \{\cdot,\cdot\}'(x) = r$ for all $\{\cdot,\cdot\}' \in \Lambda^{\mathbb{C}}$.*

3.3 Completeness conditions

Theorem 3.2 *Let $\mathfrak{H}$ be a maximal torus in a compact connected Lie group $\mathfrak{G}$. If $a \in H$ is regular, then $\mathcal{B}_a$ is a complete commutative subalgebra in $\mathbb{R}[V]^{\mathfrak{H}}$. In particular $(\mathfrak{G},\mathfrak{H})$ is an integrable pair.*

Proof. For a fixed element $a \in G$, consider the *a–bracket* defined by

$$\{f(x), g(x)\}_a = \langle a, [\nabla f(x), \nabla g(x)]\rangle, \qquad f,g : G \to \mathbb{R}.$$

It is well known that the Lie-Poisson bracket and a–bracket are compatible on G [5, 35]. By Λ_a denote the corresponding family of Poisson brackets:

$$\Lambda_a = \{\lambda_1\{\cdot,\cdot\}_G + \lambda_2\{\cdot,\cdot\}_a,\ \lambda_1,\lambda_2 \in \mathbb{R},\ \lambda_1^2+\lambda_2^2 \neq 0\}.$$

It is clear that almost all elements in V and H are regular. The Lie algebra $G^{\mathbb{C}}$ (outside the set $V^{\mathbb{C}} \cup H^{\mathbb{C}}$) is foliated into the planes $\{\lambda_1 a + \lambda_2 v,\ \lambda_1,\lambda_2 \in \mathbb{C}\}$, $a \in H^{\mathbb{C}}$, $v \in V^{\mathbb{C}}$. Thus, since the codimension of the set of singular elements in $G^{\mathbb{C}}$ is equal to 3, there are $a \in H$ and $v \in V$ such that the plane $\{\lambda_1 a + \lambda_2 v,\ \lambda_1,\lambda_2 \in \mathbb{C}\}$ does not contain any singular elements except 0.

The kernel of the bracket $\{\cdot,\cdot\}_G + \lambda\{\cdot,\cdot\}_a$ at v is $\operatorname{ann}_{G^{\mathbb{C}}}(v+\lambda a)$. Therefore all brackets in $\Lambda_a^{\mathbb{C}}$ have the maximal rank at v, equal to $\dim G - \operatorname{rank} G$. Casimir functions of the bracket $\{\cdot,\cdot\}_G + \lambda\{\cdot,\cdot\}_a$ are $p(\cdot+\lambda a)$, where p is $\mathrm{Ad}_{\mathfrak{G}}$–invariant polynomial, so $\mathcal{F}_{\Lambda_a} = \mathcal{A}_a$. From theorem 3.1 it follows that $\mathcal{A}_a$ is a complete involutive set of functions at v:

$$\dim A_a(v) = \dim \operatorname{span}\{\nabla p_a^\lambda(v),\ p \in \mathcal{A}_a\} = \frac{1}{2}(\dim G + \operatorname{rank} G).$$

Since $B_a(v) = pr_V(A_a(v))$, we get:

$$\dim B_a(v) \geq \dim A_a(v) - \dim H = \frac{1}{2}(\dim G + \operatorname{rank} G) - \dim H$$

Taking into account lemma 3.1 and remark 1.3, we see that $\mathcal{B}_a$ is a complete involutive set of $\mathrm{Ad}_{\mathfrak{H}}$–invariant functions on V.

Remark 3.1 Similar statement holds for $(\mathfrak{G}, \mathfrak{H})$, where $\mathfrak{H}$ is an arbitrary torus in $\mathfrak{G}$.

Remark 3.2 The integrability of the geodesic flows on $\mathfrak{G}/\mathfrak{H}$, where $\mathfrak{H}$ is a maximal torus, by using the argument shift method, was firstly proved by M. Bordemann [11] and later independently in [9].

We have the following nontrivial generalization:

Theorem 3.3 *Let $H = \operatorname{ann}_G(a)$ for some $a \in G$ and $\mathfrak{H}$ be the corresponding connected subgroup of $\mathfrak{G}$. Suppose that there exists generic $v \in V^{\mathbb{C}}$ such that:*

$$(C1) \qquad \dim \operatorname{ann}_{G^{\mathbb{C}}}(v + \lambda a) = \operatorname{ann}_{G^{\mathbb{C}}}(v), \quad \text{for all } \lambda \in \mathbb{C},$$
$$(C2) \qquad \dim \operatorname{ann}_{H^{\mathbb{C}}} pr_{H^{\mathbb{C}}}([ad_a^{-1} v, v]) = \dim \operatorname{ann}_{G^{\mathbb{C}}}(v).$$

Then $\mathcal{B}_a$ is a complete commutative subalgebra in $\mathbb{R}[V]^{\mathfrak{H}}$. In particular $(\mathfrak{G}, \mathfrak{H})$ is an integrable pair.

Remark 3.3 Note that the homogeneous space $\mathfrak{G}/\mathfrak{H}$ is the adjoint orbit $\mathcal{O}_{\mathfrak{G}}(a)$ of $\mathfrak{G}$–action on G. If a is regular in G then $\mathfrak{H}$ is a maximal torus and theorem 3.3 is reduced to theorem 3.2.

Proof. In order to prove the theorem we shall pass from V to the orbit space $V/\mathfrak{H}$, with respect to the natural adjoint action of $\mathfrak{H}$ on V. Denote by $[v]$ the orbit of the $\mathrm{Ad}_{\mathfrak{H}}$–action through v.

It is clear that one can consider the algebra of $\mathrm{Ad}_{\mathfrak{H}}$–invariant functions $\mathbb{R}[V]^{\mathfrak{H}}$ as the algebra $\mathbb{R}[V/\mathfrak{H}]$ of functions on the orbit space $V/\mathfrak{H}$, with respect to the reduced Lie-Poisson brackets $\{\cdot,\cdot\}'$ on $V/\mathfrak{H}$ ($f' \in \mathbb{R}[V/\mathfrak{H}]$ correspond to $f \in \mathbb{R}[V]^{\mathfrak{H}}$ in the usual way: $f'([v]) = f(v)$).

It is easy to see that the algebra $\mathbb{R}[V]^{\mathfrak{H}}$ is closed with respect to the a–bracket. That is why $\{\cdot,\cdot\}_a$ induces a natural bracket $\{\cdot,\cdot\}'_a$ on the orbit space $V/\mathfrak{H}$. The brackets $\{\cdot,\cdot\}'$ and $\{\cdot,\cdot\}'_a$ are compatible as brackets on the reduced space $V/\mathfrak{H}$. Let

$$\Lambda'_a = \{\lambda_1\{\cdot,\cdot\}' + \lambda_2\{\cdot,\cdot\}'_a,\ \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1^2 + \lambda_2^2 \neq 0\}.$$

Note that $V/\mathfrak{H}$ is not smooth. However, in a neighborhood of a generic point, it is a smooth manifold of dimension $(\dim V - \dim H + \dim \operatorname{ann}_H(v))$ and we can apply theorem 3.1. The genericity conditions on $v \in V$ (i.e., minimality of dimensions of $\operatorname{ann}_H(v)$ and $\operatorname{ann}_G(v)$) means exactly that $[v]$ is a smooth point and the bracket $\{\cdot,\cdot\}'$ has maximal rank at $[v]$.

We shall prove that under assumptions (C1) and (C2), the family of induced brackets Λ'_a satisfies the conditions of theorem 3.1, for generic $[v] \in V/\mathfrak{H}$ (see lemma's 3.3, 3.4, 3.5 and 3.6 bellow). It remains to notice that Casimir functions of the Poisson bracket $\{\cdot,\cdot\}'+\lambda\{\cdot,\cdot\}'_a$ are of the form $p(\cdot+\lambda a)$ where p is an $\mathrm{Ad}_{\mathfrak{G}}$–invariant. Therefore $\mathcal{B}_a$ is complete with respect to the reduced bracket $\{\cdot,\cdot\}'$ on the orbit space, or which is the same, $\mathcal{B}_a$ is a complete commutative subalgebra in $\mathbb{R}[V]^{\mathfrak{H}}$.

All spaces below are assumed to be complexified.

Lemma 3.3

$$\dim\big(\ker(\{\cdot,\cdot\}' + \lambda\{\cdot,\cdot\}'_a)[v]\big) = \dim pr_V(\mathrm{ann}_G(v+\lambda a)), \tag{12}$$

for generic $v \in V$.

Proof. Let $v \in V$ be generic, then the (co)tangent space to $V/\mathfrak{H}$ at $[v]$ can be naturally identified with the subspace $J \subset V$ generated by the gradients of the $\mathrm{Ad}_{\mathfrak{H}}$–invariant functions at v. It is clear that J coincides with the orthogonal complement to the tangent space of the orbit $\mathcal{O}_{\mathfrak{H}}(v)$

$$J = \mathrm{span}\,\{\nabla f(v),\ f \in \mathbb{R}[V]^{\mathfrak{H}}\} = [v,H]^{\perp}.$$

Let $K \subset J$ be the kernel of the bracket $\{\cdot,\cdot\}' + \lambda\{\cdot,\cdot\}'_a$ at $[v]$. We have

$$\begin{aligned} K &= \{\eta \in J,\ \langle v+\lambda a, [\eta, J]\rangle = 0\} \\ &= \{\eta \in J,\ \langle [v+\lambda a,\eta], [H,v]^{\perp}\rangle = 0\} \\ &= \{\eta \in J,\ [v+\lambda a,\eta] \in [H,v]\} \\ &= \{\eta \in J,\ [v+\lambda a,\eta] = [h,v],\ \text{for some } h \in H\}. \end{aligned}$$

Let us prove that $K = pr_V(\mathrm{ann}_G(v+\lambda a))$. Since $h \in H = \mathrm{ann}_G(a)$, we get $[h,v] = [h, v+\lambda a]$ and

$$K = \{\eta \in J,\ [v+\lambda a, h+\eta] = 0,\ \text{for some } h \in H\}. \tag{13}$$

Hence $h+\eta \in \mathrm{ann}_G(v+\lambda a)$. Therefore $\eta \in pr_V(\mathrm{ann}_G(v+\lambda a))$, i.e., $K \subset pr_V(\mathrm{ann}_G(v+\lambda a))$.

Let us prove the inverse inclusion $pr_V(\mathrm{ann}_G(v+\lambda a)) \subset K$. Take $\eta \in pr_V(\mathrm{ann}_G(v+\lambda a))$. Then there exists h, such that $[\eta+h, v+\lambda a] = 0$. Besides

$$\langle \eta, [v,H]\rangle = -\langle [v,\eta], H\rangle = -\langle \lambda[a,\eta] + [h,v], H\rangle = 0,$$

i.e., $\eta \in [v,H]^{\perp} = J$. From (13) we get that $\eta \in K$. This completes the proof.

Lemma 3.4 *Suppose that (C1) holds. Then*

$$\dim pr_V(\mathrm{ann}_G(v+\lambda a)) = \dim pr_V(\mathrm{ann}_G(v)), \tag{14}$$

for generic $v \in V$.

Proof. Note that

$$\mathrm{ann}_H(v+\lambda a) = \{h \in H,\ [h, v+\lambda a] = [h,v] = 0\} = \mathrm{ann}_H(v).$$

Using this identity and (C1), we have

$$\begin{aligned} &\dim pr_V(\mathrm{ann}_G(v+\lambda a)) = \dim \mathrm{ann}_G(v+\lambda a) - \dim \mathrm{ann}_H(v+\lambda a) = \\ &\dim \mathrm{ann}_G(v) - \dim \mathrm{ann}_H(v) = \dim pr_V(\mathrm{ann}_G(v)). \end{aligned}$$

Lemma is proved.

Lemma 3.5 *The corank of the reduced a-brackets is given by:*

$$\dim\ker\{\cdot,\cdot\}'_a([v]) = \dim[a,[v,H]^\perp]\cap[v,H], \qquad (15)$$

for generic $v \in V$.

Proof. As above, let $J = [v,H]^\perp$ be the (co)tangent space to $V/\mathfrak{H}$ at $[v]$. Then

$$\begin{aligned}\ker\{\cdot,\cdot\}_a{}'([v]) &= \{\eta\in J,\ \langle a,[\eta,[v,H]^\perp]\rangle = 0\}\\ &= \{\eta\in J,\ \langle [a,\eta],[v,H]^\perp]\rangle = 0\}\\ &= \{\eta\in J,\ [a,\eta]\in[v,H]\}.\end{aligned}$$

Since $\mathrm{ad}_a : V \to V$ is invertible, we have

$$\dim\{\eta\in J,\ [a,\eta]\in[v,H]\} = \dim[a,J]\cap[v,H] = \dim[a,[v,H]^\perp]\cap[v,H].$$

Lemma 3.6

$$\dim[a,[v,H]^\perp]\cap[v,H] = \dim\mathrm{ann}_H(pr_H[ad_a^{-1}v,v]) - \dim\mathrm{ann}_H(v), \qquad (16)$$

Proof. Let $h\in H$ be a vector such that $[v,h]$ belongs to $[a,[v,H]^\perp]$. Since ad_a is invertible on V, we have that

$$(\mathrm{ad}_a)^{-1}[v,h]\in[v,H]^\perp,$$

or equivalently:

$$\langle(\mathrm{ad}_a)^{-1}[v,h],[v,H]\rangle = 0. \qquad (17)$$

We can consider (17) as a linear system of $\dim H$ equations with $\dim H$ unknown variables that determines h.

Since $[a,H]=0$ we have

$$(\mathrm{ad}_a)^{-1}[v,h] = [(\mathrm{ad}_a)^{-1}v,h]. \qquad (18)$$

Using (18), the Jacobi identity and skew-symmetry of the operators ad_a, $(\mathrm{ad}_a)^{-1}$, ad_v, we can transform the left hand side of (17) as follows

$$\begin{aligned}\langle(\mathrm{ad}_a)^{-1}[v,h],[v,H]\rangle &= \langle[(\mathrm{ad}_a)^{-1}v,h],[v,H]\rangle\\ &= -\langle[v,[(\mathrm{ad}_a)^{-1}v,h]],H\rangle\\ &= \langle[h,[v,(\mathrm{ad}_a)^{-1}v]],H\rangle + \langle[(\mathrm{ad}_a)^{-1}v,[h,v]],H\rangle\end{aligned} \qquad (19)$$

Similarly, the second summand in (19) can be transformed in the following way:

$$\begin{aligned}\langle[(\mathrm{ad}_a)^{-1}v,[h,v]],H\rangle &= -\langle[h,v],[(\mathrm{ad}_a)^{-1}v,H]\rangle\\ &= -\langle[h,v],(\mathrm{ad}_a)^{-1}[v,H]\rangle\\ &= \langle(\mathrm{ad}_a)^{-1}[h,v],[v,H]\rangle\end{aligned} \qquad (20)$$

Therefore, from (19) and (20) we get:

$$2\langle(\mathrm{ad}_a)^{-1}[v,h],[v,H]\rangle = \langle[h,[v,(\mathrm{ad}_a)^{-1}v]],H\rangle$$

Thus (17) becomes

$$\langle[h,[v,(\mathrm{ad}_a)^{-1}v]],H\rangle = 0$$

which means that the space L of solutions of (17) is exactly

$$\mathrm{ann}_H(pr_H[v,(\mathrm{ad}_a)^{-1}v]).$$

By our construction $[a,[v,H]^{\perp}]\cap[v,H]$ coincides with $[v,L]$. It follows from this that

$$\dim[a,[v,H]^{\perp}]\cap[v,H]=\dim L-\dim(\mathrm{ann}_H(v)\cap L).$$

But from (17) we see immediately that $\mathrm{ann}_H(v)\subset L$. Consequently,

$$\dim[a,[v,H]^{\perp}]\cap[v,H]=\dim\mathrm{ann}_H(pr_H[ad_a^{-1}v,v])-\dim\mathrm{ann}_H(v).$$

Lemma is proved.

Thus from lemmas 3.3, 3.4 it follows that

$$\dim\left(\ker(\{\cdot,\cdot\}'+\lambda\{\cdot,\cdot\}'_a)[v]\right)=\dim pr_V(\mathrm{ann}_G(v)),$$

for generic $v\in V$ and any $\lambda\in\mathbb{C}$. From lemmas 3.5, 3.6 and the condition (C2) we have

$$\dim\ker\left(\{\cdot,\cdot\}'_a[v]\right)=\dim\mathrm{ann}_H(pr_H[ad_a^{-1}v,v])-\dim\mathrm{ann}_H(v)=$$
$$\dim\mathrm{ann}_G(v)-\dim\mathrm{ann}_H(v)=\dim pr_V(\mathrm{ann}_G(v)).$$

We have found $[v]\in V/\mathfrak{H}$ at which all the brackets have the maximal rank. Thus the conditions of theorem 3.1 are satisfied, as required.

3.4 Adjoint orbits of classical groups

Here we give an application of theorem 3.2 to the case of the Lie group $U(n)$. In fact, it can be similarly checked that the conditions of theorem 3.2 are satisfied for $SO(n)$ and $Sp(n)$ as well. We think that the same result holds for all compact groups.

Theorem 3.4 *Let $a\in\mathfrak{u}(n)$ be an arbitrary element. Let $\mathfrak{H}=\mathrm{ann}_{U(n)}(a)=\{g\in U(n),\ \mathrm{Ad}_g a=a\}$ be the isotropy subgroup. Then $\mathcal{B}_a$ is a complete involutive subalgebra in $\mathbb{R}[V]^{\mathfrak{H}}$ and consequently, $(U(n),\mathfrak{H})$ is an integrable pair.*

Proof. It suffices to prove our statement for the complexified objects. That is why instead of $U(n)$ we shall consider the complex group $GL(n,\mathbb{C})$. Without loss of generality we consider $a\in\mathfrak{gl}(n,\mathbb{C})$ of the form

$$a=\mathrm{diag}(a_1,\dots,a_n)=\mathrm{diag}(\underbrace{\alpha_1,\dots,\alpha_1}_{k_1},\underbrace{\alpha_2,\dots,\alpha_2}_{k_2},\dots,\underbrace{\alpha_r,\dots,\alpha_r}_{k_r}). \tag{21}$$

Then

$$\mathrm{ann}_{\mathfrak{gl}(n,\mathbb{C})}(a)=H=\mathfrak{gl}(k_1,\mathbb{C})+\ldots+\mathfrak{gl}(k_r,\mathbb{C}). \tag{22}$$

By theorem 3.2 we just need to verify the conditions (C1) and (C2). Since these conditions are both generic it suffices to find $v_1\in V$ for which (C1) holds and $v_2\in V$ for which (C2) holds. Then the set of $v\in V$ satisfying the both conditions simultaneously will be open and everywhere dense in V.

We shall divide the proof into two steps.

Step 1. First, let us consider the case when $k_i\leq[(n+1)/2]$, $i=1,\dots,r$.

Let us verify (C1). It is clear a can be replaced by $\tilde{a} = \mathrm{diag}(a_{p(1)}, \dots, a_{p(n)})$, where p is a permutation of $\{1, 2, \dots, n\}$, such that $a_{p(i)} \neq a_{p(i+1)}$. Let $\tilde{H} = \mathrm{ann}_{\mathfrak{gl}(n,\mathbb{C})}(\tilde{a})$, $\tilde{V} = \tilde{H}^{\perp}$. Then the (upper diagonal) matrix:

$$\tilde{v} = E_{1,2} + E_{2,3} + E_{3,4} + \dots + E_{n-1,n}$$

belongs to $\tilde{V}$ (by $E_{i,j}$ we denote the matrix with only nonzero element occupies the position (i,j)).

It can be easily seen that any element $\tilde{v} + \lambda\tilde{a}$ is regular in $\mathfrak{gl}(n,\mathbb{C})$:

$$\dim \mathrm{ann}_{\mathfrak{gl}(n,\mathbb{C})}(\tilde{v} + \lambda\tilde{a}) = \mathrm{rank}\ \mathfrak{gl}(n,\mathbb{C}),$$

which is equivalent to (C1).

To prove (C2) we again take a in the original form (21). Then $v \in V$ has the form

$$v = \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & & v_{1,r} \\ v_{2,1} & 0 & v_{2,3} & & v_{2,r} \\ v_{3,1} & v_{3,2} & 0 & & v_{3,r} \\ & & & \ddots & \\ v_{r,1} & v_{r,2} & v_{r,3} & & 0 \end{pmatrix},$$

where $v_{i,j}$ is a $k_i \times k_j$ complex matrix.

Since $H = \mathfrak{gl}(k_1,\mathbb{C}) + \dots + \mathfrak{gl}(k_r,\mathbb{C})$ we have the natural decomposition $h = h_1 + \dots + h_r$, where $h_i \in \mathfrak{gl}(k_i,\mathbb{C})$.

Then the condition that h belongs to $\mathrm{ann}_H(pr_H[v, (\mathrm{ad}_a)^{-1}v])$ is separated into the following r independent equations:

$$[h_i, \frac{1}{\alpha_i - \alpha_1} v_{i,1}v_{1,i} + \frac{1}{\alpha_i - \alpha_2} v_{i,2}v_{2,i} + \dots + \frac{1}{\alpha_i - \alpha_r} v_{i,r}v_{r,i}] = 0, \tag{23}$$

$i = 1, \dots, r$. Since $k_i \leq [(n+1)/2]$, we can choose $v \in V$ so that the matrix

$$c_i = \frac{1}{\alpha_i - \alpha_1} v_{i,1}v_{1,i} + \frac{1}{\alpha_i - \alpha_2} v_{i,2}v_{2,i} + \dots + \frac{1}{\alpha_i - \alpha_r} v_{i,r}v_{r,i}$$

is regular in $\mathfrak{gl}(k_i,\mathbb{C})$. Then $h_i \in \mathrm{ann}_{\mathfrak{gl}(k_i,\mathbb{C})}(c_i)$ and

$$\begin{aligned} \dim \mathrm{ann}_H(pr_H[v, (\mathrm{ad}_a)^{-1}v]) &= \dim \mathrm{ann}_{\mathfrak{gl}(k_1,\mathbb{C})}(c_1) + \dots + \dim \mathrm{ann}_{\mathfrak{gl}(k_r,\mathbb{C})}(c_r) \\ &= \mathrm{rank}\ \mathfrak{gl}(k_1,\mathbb{C}) + \dots + \mathrm{rank}\ \mathfrak{gl}(k_r,\mathbb{C}) = n, \end{aligned}$$

Therefore the condition (C2) holds as well.

Step 2. Suppose that there is i such that $k_i > [(n+1)/2]$. Without loss of generality we assume $i = r$ and $k_r = \sum_{i=1}^{r-1} k_i + 1 + m$, $m > 0$. Let $a = \mathrm{diag}(a', a'')$, where

$$a' = \mathrm{diag}(\underbrace{\alpha_1, \dots, \alpha_1}_{k_1}, \underbrace{\alpha_2, \dots, \alpha_2}_{k_2}, \dots, \underbrace{\alpha_r, \dots, \alpha_r}_{k_r - m}),$$

$$a'' = \mathrm{diag}(\underbrace{\alpha_r, \dots, \alpha_r}_{m}).$$

Let $H' = \mathrm{ann}_{\mathfrak{gl}(n-m,\mathbb{C})}(a')$ and let V' be the orthogonal complement of H' in $\mathfrak{gl}(n-m,\mathbb{C})$. It easy to see that each $v \in V$ is $\mathrm{Ad}_{\mathfrak{H}}$–conjugate to some element from V'. Thus, without loosing generality we can look for $v \in V'$ satisfying (C1), (C2).

Since $a' \in \mathfrak{gl}(n-m,\mathbb{C})$ satisfies the condition of step 1, for a generic element $v \in V'$ we have two relations:

$$\dim \operatorname{ann}_{\mathfrak{gl}(n-m,\mathbb{C})}(v+\lambda a') = \dim \operatorname{ann}_{\mathfrak{gl}(n-m,\mathbb{C})}(v) = n-m, \quad \lambda \in \mathbb{C},$$
$$\dim \operatorname{ann}_{H'} pr_{H'}([ad_{a'}^{-1} v, v]) = \dim \operatorname{ann}_{\mathfrak{gl}(n-m,\mathbb{C})}(v) = n-m$$

On the other side, for a generic element $v \in V'$ we have:

$$\operatorname{ann}_{\mathfrak{gl}(n,\mathbb{C})}(v+\lambda a) = \operatorname{ann}_{\mathfrak{gl}(n-m,\mathbb{C})}(v+\lambda a') + \mathfrak{gl}(m,\mathbb{C}), \quad \lambda \in \mathbb{C}$$
$$\operatorname{ann}_H(pr_H[\operatorname{ad}_a^{-1} v, v]) = \operatorname{ann}_{H'} pr_{H'}([ad_{\tilde{a}}^{-1} v, v]) + \mathfrak{gl}(m,\mathbb{C}).$$

Thus conditions (C1) and (C2) are satisfied, namely

$$\dim \operatorname{ann}_{\mathfrak{gl}(n,\mathbb{C})}(v+\lambda a) = \dim \operatorname{ann}_{\mathfrak{gl}(n,\mathbb{C})}(v) = n-m+\dim \mathfrak{gl}(m,\mathbb{C}), \quad \lambda \in \mathbb{C},$$
$$\dim \operatorname{ann}_H pr_H([ad_a^{-1} v, v]) = n-m+\dim \mathfrak{gl}(m,\mathbb{C}).$$

This completes the proof.

Remark 3.4 Just in the same way we can prove the integrability of $(U(n), U(k_1) \times U(k_2) \times \ldots \times U(k_r))$ for $k_1 + \ldots + k_r < n$. Indeed, there are commutative subalgebra $T \subset V$ and diagonal matrix a such that $\mathfrak{u}(k_1) + \ldots + \mathfrak{u}(k_r) + T = \operatorname{ann}_{\mathfrak{u}(n)}(a)$. Let $\mathbb{R}[T]$ be the algebra of polynomials on T considered as functions on V. Simple calculations show that $\mathcal{B}_a + \mathbb{R}[T]$ is a complete commutative algebra of $\operatorname{Ad}_{U(k_1)\times\ldots\times U(k_r)}$ invariants. A similar statement can be proved for the special unitary group $SU(n)$.

4 Chains of subalgebras

Trofimov and Thimm devised a method for constructing functions in involution on a Lie algebra G by using chains of subalgebras [33, 34]. Suppose we are given a chain of connected compact subgroups $\mathfrak{H} = \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \mathfrak{G}_2 \subset \ldots \subset \mathfrak{G}_n = \mathfrak{G}$, and the corresponding chain of subalgebras in G:

$$H = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n = G \tag{24}$$

Let $\pi_i : G \to G_i$ be the orthogonal projection. Let $p_1 \in \mathbb{R}[G_i]^{\mathfrak{G}_i}$, $p_2 \in \mathbb{R}[G_j]^{\mathfrak{G}_j}$. Then the Lie-Poisson bracket of $p_1 \circ \pi_i$ and $p_2 \circ \pi_j$ vanishes identically on G [33, 34].

With respect to (24) we have orthogonal decomposition of V:

$$V = V_1 + V_2 + \ldots + V_n, \tag{25}$$

where V_i is the orthogonal complement of G_{i-1} in G_i. In particular, $\pi_i(V) = V_1 + \ldots + V_i$, $i = 1, \ldots, n$.

Let

$$\mathcal{B}_i = \{f : V \to \mathbb{R}, f(v) = p(\pi_i(v)),\ p \in \mathbb{R}[G_i]^{\mathfrak{G}_i}\}, \quad i = 1, \ldots, n,$$
$$\mathcal{B} = \mathcal{B}_1 + \ldots + \mathcal{B}_n. \tag{26}$$

The following statement is obvious.

Lemma 4.1 *The algebra $\mathcal{B}$ is a commutative subalgebra in $\mathbb{R}[V]^{\mathfrak{H}}$.*

We have

$$\mathrm{ddim}\,\mathcal{B} = \dim B, \quad \text{where} \quad B = B(v) = \mathrm{span}\,\{\nabla f(v), f \in \mathcal{B}\}, \tag{27}$$

for generic $v \in V$. On the other side, by the definition of $\mathcal{B}$

$$\begin{aligned} \dim B(v) \quad &\geq \quad \dim pr_{V_1} \mathrm{span}\,\{\nabla p(\pi_1(v)),\ p \in \mathbb{R}[G_1]^{\mathfrak{G}_1}\} + \\ &+ \quad \dim pr_{V_2} \mathrm{span}\,\{\nabla p(\pi_2(v)),\ p \in \mathbb{R}[G_2]^{\mathfrak{G}_2}\} + \dots \\ &+ \quad \dim pr_{V_n} \mathrm{span}\,\{\nabla p(\pi_n(v)),\ p \in \mathbb{R}[G_n]^{\mathfrak{G}_n}\}. \end{aligned} \tag{28}$$

Let us define $r_1, \dots, r_n$ by:

$$r_i = \max_{v \in V_1 + \dots + V_i} \dim pr_{V_i} \mathrm{span}\,\{\nabla p(v),\ p \in \mathbb{R}[G_i]^{\mathfrak{G}_i}\}. \tag{29}$$

Following Bazaikin [4], the *rank of the chain* (24) is defined to be the number $R = r_1 + \dots + r_n$. By (28), the number of independent functions in $\mathcal{B}$ is greater or equal to R:

$$\mathrm{ddim}\,\mathcal{B} \geq R = r_1 + \dots + r_n. \tag{30}$$

Hence, if $R = \dim V - \frac{1}{2} \dim \mathcal{O}_{\mathfrak{G}}(v)$ then $\mathcal{B}$ is complete.

In the next theorem we give examples of some integrable pairs with complete algebras $\mathcal{B} \subset \mathbb{R}[V]^{\mathfrak{H}}$ obtained by using chains of subalgebras.

Theorem 4.1

$$\begin{array}{ll} (SO(n), SO(k_1) \times SO(k_2)), & k_1, k_2 \geq 0, k_1 + k_2 \leq n \\ (U(n), U(1)^{k_1} \times U(k_2) \times U(k_3)), & k_1, k_2, k_3 \geq 0, k_1 + k_2 + k_3 \leq n \\ (U(n), SO(k)), & k \leq n \\ (SO(n_1) \times SO(n_2), SO(k)), & k \leq n_1, n_2 \\ (U(n_1) \times U(n_2), U(k)), & k \leq n_1, n_2 \end{array}$$

are integrable pairs. In the last two examples $SO(k)$ and $U(k)$ are diagonally embedded into $SO(n_1) \times SO(n_2)$ and $U(n_1) \times U(n_2)$ respectively.

The proof is reduced to computing ranks of some natural chains of subalgebras. As an example we shall indicate the chains for the first and forth example.

$$\begin{aligned} &H = \mathfrak{so}(k_1) + \mathfrak{so}(k_2) \subset \mathfrak{so}(k_1 + k_2) \subset \mathfrak{so}(k_1 + k_2 + 1) \subset \dots \subset \mathfrak{so}(n), \\ &r_i = k_1 + i - 1, \quad i = 1, \dots, k_2 - k_1, \\ &r_{k_2 - k_1 + i} = \left[\frac{2k_2 + i}{2}\right], i = 0, 1, \dots, n - 2k_2, \quad (0 \leq k_1 \leq k_2), \end{aligned}$$

$$\begin{aligned} H = \mathfrak{so}(k) &\subset \mathfrak{so}(k) \oplus \mathfrak{so}(k) \subset \mathfrak{so}(k+1) \oplus \mathfrak{so}(k) \subset \dots \subset \mathfrak{so}(n_1) \oplus \mathfrak{so}(k) \subset \\ &\subset \mathfrak{so}(n_1) \oplus \mathfrak{so}(k+1) \subset \dots \subset \mathfrak{so}(n_1) \oplus \mathfrak{so}(n_2), \\ r_1 = \left[\frac{k}{2}\right]&, r_2 = \left[\frac{k+1}{2}\right], \dots, r_{n_1 - k + 1} = \left[\frac{n_1}{2}\right], \\ r_{n_1 - k + 2} = \left[\frac{k_1}{2}\right]&, r_{n_1 - k + 3} = \left[\frac{k+2}{2}\right], \dots, r_{n_1 + n_2 - k + 1} = \left[\frac{n_2}{2}\right]. \end{aligned}$$

The condition $R = \sum r_i = \dim V - \frac{1}{2} \dim \mathcal{O}_{\mathfrak{G}}(v)$ is straightforward.

Example 4.1 In theorem 4.1, we consider naturally embedded subgroups (as block matrices). However, the same construction can be applied to some other embeddings. As an example, let us consider the so-called Aloff–Wallach spaces $M_{k,l} = SU(3)/T_{k,l}$ [1], where

$$T_{k,l} = \left\{ \begin{pmatrix} e^{2\pi ik\theta} & 0 & 0 \\ 0 & e^{2\pi il\theta} & 0 \\ 0 & 0 & e^{-2\pi i(k+l)\theta} \end{pmatrix}, \ \theta \in \mathbb{R} \right\}, k, l \in \mathbb{Z}, |k| + |l| \neq 0, kl > 0.$$

Among spaces $M_{k,l}$ there are infinitely many with different cohomological structures: if k, l are relatively prime, then $H^4(M_{k,l}, \mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$, with $r = |k^2 + l^2 + kl|$ [1].

Consider the following chain of subgroups

$$\mathfrak{G}_0 = T_{k,l} \subset \mathfrak{G}_1 \subset \mathfrak{G}_2 = SU(3),$$

where

$$\mathfrak{G}_1 = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det g^{-1} \end{pmatrix} \in SU(3), \ g \in U(2) \right\} \cong U(2).$$

Let $V = V_1 + V_2$ be the orthogonal decomposition as above

$$V_1 = \left\{ \begin{pmatrix} ia & z_{12} & 0 \\ -\bar{z}_{12} & ib & 0 \\ 0 & 0 & -i(a+b) \end{pmatrix}, \ ka + lb + (k+l)(a+b) = 0 \right\},$$

$$V_2 = \left\{ \begin{pmatrix} 0 & 0 & z_{13} \\ 0 & 0 & z_{23} \\ -\bar{z}_{13} & -\bar{z}_{23} & 0 \end{pmatrix} \right\}.$$

Aloff and Wallach proved that the $SU(3)$–invariant Riemannian metrics ds_t^2 on $M_{k,l}$ obtained from the quadratic forms

$$B_t(v, v) = (1+t)\langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle, \quad v = v_1 + v_2, \ v_i \in V_i, \quad -1 < t < 0$$

have positive sectional curvature.

The algebra of functions $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ is complete. Indeed, $\mathcal{B}$ is generated by 4 independent functions:

$$f_1 = a + b, \quad f_2 = \langle v_1, v_1 \rangle, f_3 = \langle v_1 + v_2, v_1 + v_2 \rangle, f_4 = tr(v_1 + v_2)^3,$$

and $4 = \dim V - \frac{1}{2}\mathcal{O}_{\mathfrak{G}}(v) = 7 - 3$.

Since $B_t(v, v) \in \mathcal{B}$, the geodesic flows of metrics ds_t^2 are completely integrable.

5 Generalized chain method

Let θ be a Cartan involution on G and let $G = L + W$ be the corresponding orthogonal decomposition into the eigen-spaces of θ. Then (G, L) is called a *symmetric pair* and the following relations hold:

$$[L, L] \subset L, \quad [L, W] \subset W, \quad [W, W] \subset L.$$

Consider the following algebra of polynomials on G:

$$\mathcal{A}_{L,W} = \{f(x) = p(\lambda l + w), \ p \in \mathbb{R}[G]^{\mathfrak{G}}, \ \lambda \in \mathbb{R}\}, \tag{31}$$

where $x = l + w$ is the orthogonal decomposition of $x \in G$ ($l \in L$, $w \in W$).

Let $\mathbb{R}[L]$ be the algebra of polynomials on L lifted to the polynomials on G.

Theorem 5.1 *(i) $\mathcal{A}_{L,W}$ is a commutative algebra of polynomials in involution with polynomials from $\mathbb{R}[L]$, i.e., commutative subalgebra in $\mathbb{R}[G]^{\mathfrak{L}}$.*

(ii) $\mathcal{A}_{L,W} + \mathbb{R}[L]$ is a complete algebra of polynomials on G. In particular, if $\mathcal{A}_L$ is a complete commutative subalgebra of $\mathbb{R}[L]$, then $\mathcal{A}_{L,W} + \mathcal{A}_L$ will be a complete commutative algebra on G.

This result was first proof by Mikityuk [23]. Also, theorem 5.1 is a special case of theorem 1.5 [5]. It is related to the compatibility of the Lie-Poisson bracket $\{\cdot,\cdot\}_G$ and the *θ–bracket* defined by:

$$\{f,g\}_\theta(x) = \langle x, [\nabla f(x), \nabla g(x)]_\theta\rangle,$$

where $[\cdot,\cdot]_\theta$ is a new operation on G which differs from the standard one $[\cdot,\cdot]$ by the only property that W is assumed to be commutative

$$[l_1 + w_1, l_2 + w_2]_\theta = [l_1, l_2] + [l_1, w_2] + [w_1, l_2]$$

(for more details and related references see [5]).

Let a homogeneous space $\mathfrak{G}/\mathfrak{H}$ be such that $\mathfrak{H} \subset \mathfrak{L}$, where $(\mathfrak{G}, \mathfrak{L})$ is a symmetric pair. Then V has the orthogonal decomposition $V = U + W$, where $L = H + U$, $G = L + W$.

Let $\mathcal{B}_{U,W}$ be the restriction of $\mathcal{A}_{L,W}$ to $V = U + W$:

$$\mathcal{B}_{U,W} = \{f(v) = p(\lambda u + w),\ p \in, \mathbb{R}[G]^{\mathfrak{G}}\ \lambda \in \mathbb{R}\}, \tag{32}$$

$v = u + w$ is the orthogonal decomposition of $v \in V$ ($u \in U$, $w \in W$).

By $\mathbb{R}[U]^{\mathfrak{H}}$ and $\mathbb{R}[V]^{\mathfrak{H}}$ we shall denote the algebras of $\mathrm{Ad}_{\mathfrak{H}}$ invariants on U and V respectively. The polynomials from $\mathbb{R}[U]^{\mathfrak{H}}$ can be naturally lifted to V. From the first part of theorem 6.1 we get

Lemma 5.1 *$\mathcal{B}_{U,W}$ is a commutative subalgebra of $\mathbb{R}[V]^{\mathfrak{H}}$. All elements of $\mathcal{B}_{U,W}$ are in involution with the polynomials from $\mathbb{R}[U]^{\mathfrak{H}}$.*

The proof of the second part of theorem 6.1 given in [35] leads to the following sufficient condition for the completeness of the algebra $\mathcal{B}_{U,W} + \mathbb{R}[U]^{\mathfrak{H}}$ (see pages 232-237, 252-253).

Theorem 5.2 *Suppose that there exist $u \in U$, $w \in W$ such that:*

(D1) $\lambda u + w$ is a regular element in $G^{\mathbb{C}}$, for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$;

(D2) u is a regular element in $L^{\mathbb{C}}$;

(D3) the orthogonal projection of u to $\mathrm{ann}_{L^{\mathbb{C}}}(w)$ is a regular element in $\mathrm{ann}_{L^{\mathbb{C}}}(w)$.

Then $\mathcal{B}_{U,W} + \mathbb{R}[U]^{\mathfrak{H}}$ is a complete subalgebra of $\mathbb{R}[V]^{\mathfrak{H}}$. If $\mathcal{B}_U$ is a complete commutative subalgebra of $\mathbb{R}[U]^{\mathfrak{H}}$, then $\mathcal{B}_{U,W} + \mathcal{B}_U$ is a complete commutative subalgebra of $\mathbb{R}[V]^{\mathfrak{H}}$.

Proof. As a particular case of theorem 8, page 234 [35], one can obtain that under the assumptions of theorem 5.2, the algebra $\mathcal{A}_{L,W} + \mathbb{R}[L]$ is complete at a point $u + w$. Moreover, the skew–orthogonal complement to the vector space $A = \mathrm{span}\{\nabla f(u + w),\ f \in \mathcal{A}_{L,W}\}$ with respect to the Lie-Poisson bracket is $A + L$.

Now, one can prove the similar result for the reduced bracket, namely: the skew–orthogonal complement in $J = [v, H]^\perp$ (space spanned by the gradients of functions $f \in \mathbb{R}[V]^{\mathfrak{H}}$) to the vector space $B = \mathrm{span}\{\nabla f(v),\ f \in \mathcal{B}_{U,W}\}$ is $B + [u, H]^\perp$ ($[u, H]^\perp$ is generated by the gradients of functions in $\mathbb{R}[U]^{\mathfrak{H}}$).

The completeness of the algebra $\mathcal{B}_{U,W} + \mathbb{R}[U]^{\mathfrak{H}}$ now follows from the following general fact. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two algebras of functions on a Poisson manifold $(M, \{\cdot,\cdot\}_M)$ that commute $\{\mathcal{F}_1, \mathcal{F}_2\}_M = 0$. Suppose $\mathcal{F}_1$ is commutative. If the skew–orthogonal complement to $\operatorname{span}\{df(x),\ f \in \mathcal{F}_1\} \subset T_x^*M$ with respect to the Poisson bracket (regarded as a bilinear form on T_x^*M) is $\operatorname{span}\{df(x),\ f \in \mathcal{F}_1\} + \operatorname{span}\{df(x),\ f \in \mathcal{F}_2\}$ then the algebra $\mathcal{F}_1 + \mathcal{F}_2$ is complete at x.

Remark 5.1 The conditions (D1) and (D3) of theorem 5.2 are automatically satisfied in the following situation. Suppose that $(\mathfrak{G}, \mathfrak{L})$ is a maximal rank symmetric space. Then a generic $w \in W$ is regular and $\operatorname{ann}_{\mathfrak{L}}(w)$ is equal to zero. Thus (D1) and (D3) hold. Thus, if $(\mathfrak{L}, \mathfrak{H})$ is an integrable pair such that (D2) holds, then from theorem 5.2 we get that $(\mathfrak{G}, \mathfrak{H})$ is also an integrable pair.

Suppose we are given a chain of connected subgroups $\mathfrak{H} = \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \ldots \subset \mathfrak{G}_n = \mathfrak{G}$ and the corresponding chain of subalgebras $H = G_0 \subset G_1 \subset \ldots \subset G_n = G$. As above we have the orthogonal decomposition $V = V_1 + V_2 + \ldots + V_n$ such that

$$G_i = H + V_1 + \ldots + V_i, \quad i = 1, \ldots, n.$$

Suppose that either (G_i, G_{i-1}) is a symmetric pair or V_i is a subalgebra of G, for all $i = 1, \ldots, n$.

For symmetric pairs (G_i, G_{i-1}), let $\mathcal{B}_i$ be the algebra of polynomials on V defined by:

$$\mathcal{B}_i = \{f(v) = p(\lambda(v_1 + \ldots + v_{i-1}) + v_i),\ p \in \mathbb{R}[G_i]^{\mathfrak{G}_i},\ \lambda \in \mathbb{R}\}, \tag{33}$$

where v_i is the orthogonal projection of $v \in V$ to V_i.

Otherwise, if V_i is a subalgebra, then let $\mathcal{B}_i$ be an arbitrary complete involutive set of polynomials on V_i lifted to V (obviously, these polynomials are $\mathrm{Ad}_{\mathfrak{H}}$–invariants in involution with functions defined on G_{i-1}, since in this case $G_i = G_{i-1} \oplus V_i$).

By induction, using lemma 5.1, we get the following statement:

Lemma 5.2 *$\mathcal{B} = \mathcal{B}_1 + \ldots + \mathcal{B}_n$ is a commutative subalgebra of $\mathbb{R}[V]^{\mathfrak{H}}$.*

Example 5.1 Using this method, we can extend the class of examples of integrable pairs obtained above. In particular, we can get the integrability of

$$(SO(n), SO(k_1) \times \ldots \times SO(k_r)),$$
$$(SU(n), SO(k_1) \times \ldots \times SO(k_r)),$$
$$(Sp(n), U(k_1) \times \ldots \times U(k_r)),$$

for values of parameters $k_1, \ldots, k_r, n$ described below.

In the first case we take those pairs which can be obtained as follows. Suppose that $(SO(n_1), \mathfrak{H}_1)$ and $(SO(n_2), \mathfrak{H}_2)$ are integrable pairs, where $\mathfrak{H}_1 = SO(k_1) \times \ldots \times SO(k_{r_1})$ and $\mathfrak{H}_2 = SO(l_1) \times \ldots \times SO(l_{r_2})$. Suppose that generic $u_1 \in U_1$ and $u_2 \in U_2$ are regular in $\mathfrak{so}(n_1)$ and $\mathfrak{so}(n_2)$ respectively (U_i is the orthogonal complement of H_i in $\mathfrak{so}(n_i)$, $i = 1, 2$). Then, if $SO(n_1 + n_2)/SO(n_1) \times SO(n_2)$ is a maximal rank symmetric space ($n_1 = n_2 \pm 0, 1$), by remark 5.1, $(SO(n_1 + n_2), SO(k_1) \times \ldots \times SO(k_{r_1}) \times SO(l_1) \times \ldots \times SO(l_{r_2}))$ is an integrable pair.

Further, for the second case, we take integrable pair $(SO(n), SO(k_1) \times \ldots \times SO(k_r))$ obtained by the above construction. Then $(SU(m), SO(k_1) \times \ldots \times SO(k_r))$, $m \geq n$ will be also an integrable pair. Since $SU(n)/SO(n)$ is a maximal rank symmetric space, by remark 5.1, it is clear that $(SU(n), SO(k_1) \times \ldots \times SO(k_r))$ is an integrable pair.

For $m > n$, we can just add the polynomials given by the usual chain method and the natural filtration $SU(n) \subset U(n) \subset U(n+1) \subset \ldots \subset SU(m)$.

In the last case, we take pairs $(Sp(n), U(k_1) \times \ldots \times U(k_r))$, such that $(k_i \leq [(n+1)/2],\ i = 1, \ldots, r)$. We can use remark 5.1 for a maximal rank symmetric space $Sp(n)/U(n)$.

6 Sub-Riemannian structures on $\mathfrak{G}/\mathfrak{H}$

6.1 Sub-Riemannian geodesic flows

The above constructions can be used for obtaining $\mathfrak{G}$–invariant nonholonomic distributions and sub-Riemannian structures with completely integrable geodesic flows on homogeneous spaces $\mathfrak{G}/\mathfrak{H}$. Similar constructions on compact Lie groups are given in [15]. Sub-Riemannian geodesic flows from the point of view of integrability are also studied in [32, 17].

The distribution $\mathcal{D} \subset TQ$ is called *completely nonholonomic* (or bracket generating) if the algebra generated by sections of $\mathcal{D}$ via commutations coincides with the whole algebra of vector fields on Q. The piecewise smooth curve $\gamma(t)$ on Q is called *horizontal* (or admissible) if $\dot\gamma(t)$ belongs to $\mathcal{D}_{\gamma(t)}$ whenever it exists. By the Chow-Rashevski theorem, if the distribution is completely nonholonomic any two points of the connected manifold Q can be jointed by a horizontal curve.

The sub-Riemannian structure on a manifold Q is a Riemannian structure g given on a completely nonholonomic distribution $\mathcal{D} \subset TQ$ (see [30]). Locally shortest horizontal curves are called *sub-Riemannian geodesic lines.*

We can define the mapping $G_q : T_q^*Q \to T_qQ$ [32] such that $\mathcal{D}_q = G_q(T_q^*Q)$ and

$$p(\xi) = g_q(\xi, G_q(p)), \quad \text{for every} \quad \xi \in \mathcal{D}_q,\ p \in T_q^*Q.$$

Then the distribution and sub-Riemannian structure are both determined by the Hamiltonian function $H(p,q) = \frac{1}{2}p(G_q(p)),\ p \in T_qQ$. The projections of solutions of the corresponding Hamiltonian equations on T^*Q are sub-Riemannian geodesic lines. These projections are called *normal geodesic lines.* Most of the geodesics are normal. The geodesic lines that are not projections of these solutions are called *abnormal geodesic lines.* The Hamiltonian flow of $H = \frac{1}{2}p(G_q(p))$ is called *sub-Riemannian geodesic flow.*

Now, we turn back to the homogeneous spaces.

Definition 6.1 Suppose that $h_\phi(v) = \frac{1}{2}\langle\phi(v), v\rangle$ is an $\mathrm{Ad}_{\mathfrak{H}}$–invariant functions, where $\phi : V \to V$ is some positive, degenerate linear operator. Let $D^\phi = \phi(V)$. Then by $\mathcal{D}^\phi$ we shall denote the $\mathfrak{G}$–invariant distribution of $T(\mathfrak{G}/\mathfrak{H})$ given by:

$$\mathcal{D}^\phi_{\pi(g)} = gD^\phi \subset T_{\pi(g)}(\mathfrak{G}/\mathfrak{H}).$$

If the distribution $\mathcal{D}^\phi$ is completely nonholonomic, then by ds^2_ϕ we shall denote the corresponding sub-Riemannian metric determined by the Hamiltonian function $\tau_2(h_\phi)$.

The distribution $\mathcal{D}^\phi$ is completely nonholonomic if and only if D^ϕ and H generate the Lie algebra G by commutations.

6.2 Argument shift method

Let $H = \mathrm{ann}_G(a)$ for some $a \in G$ and let $\mathfrak{H}$ be the corresponding connected subgroup of $\mathfrak{G}$. Consider the orthogonal decomposition $G = H + V$, an element b that belongs to the center of $\mathrm{ann}_G(a)$ and define the operator $\phi_{a,b} : V \to V$ (so-called *sectional operator* [35]) by

$$\phi_{a,b}|_V = \mathrm{ad}_a^{-1} \circ \mathrm{ad}_b$$

such that $\ker \phi_{a,b} \neq 0$. For compact groups, among sectional operators we can choose positive ones. Take such $\phi_{a,b}$.

As above, let $\mathcal{B}_a$ be the algebra of functions on V obtained by shifting the argument of invariant polynomials. It easily follows from [20] that the function $h_{\phi_{a,b}}(v) = \frac{1}{2}\langle \phi_{a,b}(v), v\rangle$ belongs to $\mathcal{B}_a$. In particular, $h_{\phi_{a,b}}$ is $\mathrm{Ad}_{\mathfrak{H}}$–invariant, which allows us to construct the $\mathfrak{G}$–invariant distribution $\mathcal{D}^{\phi_{a,b}}$ of the tangent bundle $T(\mathfrak{G}/\mathfrak{H})$.

Suppose that $\mathcal{D}^{\phi_{a,b}}$ is completely nonholonomic. Then $\mathcal{F}_1 + \tau_2(\mathcal{B}_a)$ is the algebra of first integrals of the sub-Riemannian geodesic flow of the metric $ds^2_{\phi_{a,b}}$ on the homogeneous space $\mathfrak{G}/\mathfrak{H}$.

Example 6.1 Consider the flag manifold $SU(n)/\mathbb{T}^{n-1}$ ($\mathbb{T}^{n-1} = S(U(1))^n$ is the maximal torus). Let us take

$$\begin{aligned} a &= \mathrm{diag}(a_1, a_2, \dots, a_n), \ a_i \neq a_j, i \neq j, \\ b &= \mathrm{diag}(\underbrace{b_1, \dots, b_1}_{k}, \underbrace{b_2, \dots, b_2}_{n-k}). \end{aligned}$$

Then $D^{\phi_{a,b}}$ is the orthogonal complement of $\mathfrak{s}(\mathfrak{u}(k) + \mathfrak{u}(n-k))$ in $\mathfrak{su}(n)$. The sub-Riemannian space $(SU(n)/\mathbb{T}^{n-1}), \mathcal{D}^{\phi_{a,b}}, ds^2_{\phi_{a,b}})$ is a sub-Riemannian symmetric space in the terminology of Strichartz [20]. According to theorems 1.1, 3.2 the corresponding sub-Riemannian geodesic flow is completely integrable.

6.3 Chains of subalgebras

Let us given a chain of connected subgroups $\mathfrak{H} = \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \dots \subset \mathfrak{G}_n = \mathfrak{G}$ and the corresponding chain of subalgebras $H = G_0 \subset G_1 \subset \dots \subset G_n = G$. As above, we have the orthogonal decomposition $V = V_1 + V_2 + \dots + V_n$. By v_i we shall denote the orthogonal projection of $v \in V$ to V_i.

Let $\mathcal{B} = \mathcal{B}_1 + \dots + \mathcal{B}_n$ be the commutative algebra of polynomials on V given by the chain (or generalized chain) method.

Let $\phi : V \to V$ be the operator defined by:

$$\phi(v) = s_1 v_1 + \dots + s_n v_n, \quad s_1, \dots, s_n \geq 0, \ s_1 s_2 \cdots s_n = 0. \tag{34}$$

It can be easily verified that the function $h^\phi(v) = \frac{1}{2}\langle \phi(v), v\rangle$ belongs to $\mathcal{B}$ and that is $\mathrm{Ad}_{\mathfrak{H}}$ invariant.

Suppose that $\mathcal{D}^\phi$ is a completely nonholonomic distribution. Then $\mathcal{F}_1 + \tau_2(\mathcal{B})$ is the algebra of first integrals of the sub-Riemannian geodesic flow of the metric ds^2_ϕ on the homogeneous space $\mathfrak{G}/\mathfrak{H}$.

Example 6.2 Consider the Stiefel manifold $SO(n+k)/SO(k)$. We have the following natural chains of subgroups and subalgebras:

$$\mathfrak{G}_0 = SO(k) \subset \mathfrak{G}_1 = SO(k+1) \subset \dots \subset \mathfrak{G}_n = SO(k+n)$$

$$G_0 = \mathfrak{so}(k) \subset G_1 = \mathfrak{so}(k+1) \subset \ldots \subset G_n = \mathfrak{so}(k+n).$$

Let ϕ be of the form (34) such that $s_1 = \ldots = s_r = 0$, $s_{r+1}, s_{r+2}, \ldots, s_n > 0$. Then

$$D^\phi = V_{r+1} + \ldots + V_n,$$

and the distribution $\mathcal{D}^\phi$ is completely nonholonomic. From theorems 1.1 and 4.1 we get that the sub-Riemannian geodesic flow on the sub-Riemannian space $(SO(n+k)/SO(k), \mathcal{D}^\phi, ds^2_\phi)$ is completely intgerable.

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