Magnetic Geodesic Flows on Coadjoint Orbits [*†‡]

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Abstract

We describe a class of completely integrable G-invariant magnetic geodesic flows on (co)adjoint orbits of a compact connected Lie group G with magnetic field given by the Kirillov-Konstant 2-form.

1 Introduction

Let Q be a smooth manifold with a local coordinate system $x^1, \dots, x^n$ and Riemannian metric $g = (g_{ij})$. The inertial motion of the unit mass particle under the influence of the additional magnetic field given by a closed 2-form

$$\Omega = \sum_{1 \le i < j \le n} F_{ij}(x) dx^i \wedge dx^j,$$

is described by the following equations on the phase space T^*Q:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} + \sum_{j=1}^{n} F_{ij} \frac{\partial H}{\partial p_j}, \tag{1}$$

where $p_j = g_{ij}\dot{x}^i$ are canonical momenta and the Hamiltonian is $H(x,p) = \frac{1}{2}\sum g^{ij} p_i p_j$. Here g^{ij} are the coefficients of the tensor inverse to the metric.

The equations (1) are Hamiltonian with respect to the symplectic form $\omega + \rho^*\Omega$, where $\omega = \sum dp_i \wedge dx^i$ is the canonical symplectic form on T^*Q and

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$\rho : T^*Q \to Q$ is the natural projection. Namely, the corresponding Poisson bracket is given by

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i} \right) + \sum_{i,j=1}^{n} F_{ij}\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j}, \tag{2}$$

and the Hamiltonian equations $\dot{f} = \{f, H\}$ read (1). The flow (1) is called *magnetic geodesic flow* on the Riemannian manifold (Q, g) with respect to the magnetic field Ω.

In this paper we consider G-invariant magnetic geodesic flows on (co)adjoint orbits $\mathcal{O}$ of a compact connected Lie group G, where Ω is the Kirillov-Konstant 2-form (Theorem 1). The non-commutative integrability of the systems, for the normal metrics, is proved recently by Efimov [5, 6]. Following [3], we give a new, short proof of the non-commutative integrability (Theorem 2). In addition, the usual Liouville integrability by means of commuting analytic integrals is shown. One can use the commuting integrals in order to deform the normal metric to a certain class of G-invariant metrics on $\mathcal{O}$ with completely integrable magnetic geodesic flows as well (Theorem 3).

2 Magnetic Coadjoint Orbits

Let G be a compact connected Lie group with the Lie algebra $\mathfrak{g} = T_e G$. Let us fix some Ad_G-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. By the use of $\langle \cdot, \cdot \rangle$ we identify $\mathfrak{g}^*$ with $\mathfrak{g}$.

Consider the adjoint action of G and the G-orbit $\mathcal{O}(a) = \mathrm{Ad}_G(a)$ through an element $a \in \mathfrak{g}$. Let $\xi \in \mathfrak{g}$ and $x = \mathrm{Ad}_g(a)$. Since

$$\xi_x = \frac{d}{ds}\mathrm{Ad}_{\exp(s\xi)}(x)|_{s=0} = [\xi, x], \tag{3}$$

the tangent space $T_x\mathcal{O}(a)$ is simply $[\mathfrak{g}, x]$, i.e., it is the orthogonal complement to $\mathrm{ann}(x) = \{\eta \in \mathfrak{g} \,|\, [\eta, x] = 0\}$. By definition, the Kirillov-Konstant symplectic form Ω on $\mathcal{O}(a)$ is a G-invariant form, given by

$$\Omega(\eta_1, \eta_2)|_x = -\langle x, [\xi_1, \xi_2]\rangle, \quad \eta_i = [\xi_i, x], \quad i = 1, 2. \tag{4}$$

Similarly, the scalar product $\langle \cdot, \cdot \rangle$ induces the *normal metric* K_0 on $\mathcal{O}(a)$ as follows

$$K_0(\eta_1, \eta_2)|_x = \langle \xi_1, \xi_2 \rangle, \quad \eta_i = [\xi_i, x], \quad i = 1, 2. \tag{5}$$

The cotangent bundle $T^*\mathcal{O}(a)$ can be realised as a submanifold of $\mathfrak{g} \times \mathfrak{g}$

$$T^*\mathcal{O}(a) = \{(x, p) \,|\, x = \mathrm{Ad}_g(a), p \in \mathrm{ann}(x)^\perp\},$$

with the paring between $p \in T^*_x\mathcal{O}(a)$ and $\eta \in T_x\mathcal{O}(a)$ given by $p(\eta) = \langle p, \eta \rangle$. Then the canonical symplectic form ω on $T^*\mathcal{O}(a)$ can be seen as a restriction of the canonical linear symplectic form of the ambient space $\mathfrak{g}\times\mathfrak{g}$: $\sum_{i=1}^{\dim\mathfrak{g}} dp_i\wedge dx_i$, where p_i, x_i are coordinates of p and x with respect to some base of $\mathfrak{g}$.

The G-action

$$g\cdot(x,p)=(\mathrm{Ad}_g\, x, \mathrm{Ad}_g\, p) \tag{6}$$

is Hamiltonian on $(T^*\mathcal{O}(a),\omega)$. From (3) we find that the momentum mapping is given by the relation $\langle\Phi_0(x,p),\xi\rangle=\langle p,\xi_x\rangle=\langle p,[\xi,x]\rangle$. That is

$$\Phi_0(x,p)=[x,p].$$

Following Efimov [5, 6], we consider magnetic geodesic flows on $\mathcal{O}(a)$ with respect to the magnetic fields $\epsilon\Omega$, where Ω is Kirillov-Konstant 2-form (4) and $\epsilon\in\mathbb{R}$. According to (1), the adding of magnetic field $\epsilon\Omega$ to the system reflects as a perturbation of the system in p-variable by the magnetic force Π_ϵ, determined by $\langle\Pi_\epsilon,\eta\rangle=-\epsilon\langle x,[\mathrm{ad}_x^{-1}\eta,\mathrm{ad}_x^{-1}\dot{x}]\rangle$, $\eta\in T_x\mathcal{O}(a)$. Hence $\Pi_\epsilon=-\epsilon\mathrm{ad}_x^{-1}\dot{x}$.

The G-action (6) is Hamiltonian on $(T^*\mathcal{O}(a),\omega+\epsilon\Omega)$ as well [6, 7]. In our notation we have that the momentum mapping reads

$$\Phi_\epsilon(x,p)=\Phi_0(x,p)+\epsilon x=[x,p]+\epsilon x.$$

G-Invariant Magnetic Geodesic Flows. The G-invariant metrics on $\mathcal{O}(a)$ are in one-to-one correspondence with Ad_{G_a}-invariant positive definite operators

$$\varphi:\mathfrak{v}\to\mathfrak{v},\quad \mathrm{Ad}_g\circ\varphi=\varphi\circ\mathrm{Ad}_g,\quad g\in G_a,$$

where $\mathfrak{v}=T_a\mathcal{O}(a)=\mathrm{ann}(a)^\perp$ and G_a is the isotropy group of a. Namely, for a given φ, we define

$$\varphi_x=\mathrm{Ad}_g\circ\varphi\circ\mathrm{Ad}_{g^{-1}}:T_x\mathcal{O}(a)\to T_x\mathcal{O}(a),\quad x=\mathrm{Ad}_g(a),$$

and a G-invariant metric $K_\varphi(\eta_1,\eta_2)|_x=\langle\varphi_x\eta_1,\eta_2\rangle$. After Legendre transformation $T\mathcal{O}(a)\to T^*\mathcal{O}(a)$ with respect to K_φ, we get the Hamiltonian function for the given metric:

$$H_\varphi(x,p)=\frac{1}{2}\langle\varphi_x^{-1}p,p\rangle.$$

Theorem 1 *The equations of the magnetic geodesic flow on $(\mathcal{O}(a),K_\varphi)$ with respect to the magnetic term $\epsilon\Omega$, in redundant variables (x,p), are given by*

$$\dot{x}=\varphi_x^{-1}p, \tag{7}$$
$$\dot{p}=\mathrm{ad}_x^{-1}[p,\varphi_x^{-1}p]-\mathrm{pr}_{\mathrm{ann}(x)}[\mathrm{ad}_x^{-1}\varphi_x^{-1}p,p]-\epsilon\mathrm{ad}_x^{-1}\varphi_x^{-1}p. \tag{8}$$

In particular, the magnetic flow of the normal metric (5) reads

$$\dot{x}=[[x,p],x], \tag{9}$$
$$\dot{p}=[[x,p],p]+\epsilon[x,p], \tag{10}$$

Proof. The equation (7) is just the inverse of the Legendre transformation. We can derive (8) simply by using the conservation of the momentum mapping Φ_ϵ for G-invariant Hamiltonians. We have

$$\begin{aligned} \frac{d}{dt}\Phi_\epsilon(x,p) &= [\dot{x},p]+[x,\dot{p}]+\epsilon\dot{x}=0 \\ &= [\varphi_x^{-1}p,p]+[x,\dot{p}]+\epsilon[x,\mathrm{ad}_x^{-1}\varphi_x^{-1}p]=0. \end{aligned} \tag{11}$$

Since φ^{-1} is Ad_{G_a}-invariant, the term $[\varphi_x^{-1}p,p]$ belongs to $\mathrm{ann}(x)^\perp$. Thus from (11) we get

$$\mathrm{pr}_{\mathrm{ann}(x)^\perp}\dot{p}=\mathrm{ad}_x^{-1}[p,\varphi_x^{-1}p]-\epsilon\mathrm{ad}_x^{-1}\varphi_x^{-1}p. \tag{12}$$

In order to find $\mathrm{pr}_{\mathrm{ann}(x)}\dot{p}$, take the (local) orthonormal base $e_1(x),\dots,e_r(x)$ of $\mathrm{ann}(x)$. Then $\mathrm{pr}_{\mathrm{ann}(x)}\dot{p}$ is determined from the condition that the trajectory $(x(t),p(t))$ satisfies constraints

$$\frac{d}{dt}\langle p,e_i(x)\rangle=\langle\dot{p},e_i(x)\rangle+\langle p,\dot{e}_i(x)\rangle=0,\quad i=1,\dots,r. \tag{13}$$

From $[e_i(x),x]\equiv 0$, $i=1,\dots,r$, we get

$$[\dot{e}_i(x),x]+[e_i(x),\dot{x}]=[\dot{e}_i(x),x]+[e_i(x),[x,\mathrm{ad}_x^{-1}\varphi_x^{-1}p]]=0\quad i=1,\dots,r. \tag{14}$$

Furthermore, combining (14) and the Jacobi identities

$$[e_i,[x,\mathrm{ad}_x^{-1}\varphi_x^{-1}p]]+[x,[\mathrm{ad}_x^{-1}\varphi_x^{-1}p,e_i]]+[\mathrm{ad}_x^{-1}\varphi_x^{-1}p,[e_i,x]]=0,\quad i=1,\dots,r$$

we obtain $\dot{e}_i(x)=[e_i(x),\mathrm{ad}_x^{-1}\varphi_x^{-1}p]$ (modulo $\mathrm{ann}(x)$). Whence, using (13) we get $\langle\dot{p},e_i(x)\rangle+\langle[\mathrm{ad}_x^{-1}\varphi_x^{-1}p,p],e_i\rangle=0$, $i=1,\dots,r$, i.e,

$$\mathrm{pr}_{\mathrm{ann}(x)}\dot{p}=-\mathrm{pr}_{\mathrm{ann}(x)}[\mathrm{ad}_x^{-1}\varphi_x^{-1}p,p]. \tag{15}$$

The relations (12) and (15) proves (8).

Now, for the normal metric K_0 we have $\varphi_x=-\mathrm{ad}_x^{-1}\circ\mathrm{ad}_x^{-1}$ and the Hamiltonian is

$$H_0=-\frac{1}{2}\langle\mathrm{ad}_x\mathrm{ad}_x p,p\rangle=\frac{1}{2}\langle[x,p],[x,p]\rangle=\frac{1}{2}\langle\Phi_0(x,p),\Phi_0(x,p)\rangle. \tag{16}$$

The equation (9) follows directly from (7), while (12) and (15) become

$$\begin{aligned} &\mathrm{pr}_{\mathrm{ann}(x)^\perp}\dot{p}=\mathrm{ad}_x^{-1}[p,[x,[p,x]]]+\epsilon[x,p], \\ &\mathrm{pr}_{\mathrm{ann}(x)}\dot{p}=\mathrm{pr}_{\mathrm{ann}(x)}[[x,p],p]. \end{aligned}$$

Again, the Jacobi identity gives

$$[p,[x,[p,x]]=[x,[[x,p],p]]=\mathrm{ad}_x(\mathrm{pr}_{\mathrm{ann}(x)^\perp}[[x,p],p])$$

which together with the above formulae proves (10). $\square$

The geometry of the Hamiltonian flows on cotangent bundles, in this representation, is studied by Bloch, Brockett and Crouch [1]. The system (9), (10), for $\epsilon = 0$, agrees with the equations (2.7) given in [1], while the system (7), (8) differs from the equations (2.19) [1]. The equations (2.19) [1] describe the geodesic flows of submersion (or collective) metrics on the orbit $\mathcal{O}(a)$, and, in general, are not G-invariant. Recall that the submersion metrics are given by Hamiltonians of the form $H = \frac{1}{2}\langle \Phi_0(x,p), \phi\, \Phi_0(x,p)\rangle$, where ϕ is a symmetric, positive definite operator on $\mathfrak{g}$. Specially, K_0 is both G-invariant and submersion metric.

3 Integrable Flows

Let $\mathcal{F}_1^\epsilon$ be the algebra of all analytic, polynomial in momenta, functions of the form $\mathcal{F}_1^\epsilon = \{p \circ \Phi_\epsilon \,|\, p \in \mathbb{R}[\mathfrak{g}]\}$ and $\mathcal{F}_2$ be the algebra of all analytic, polynomial in momenta, G-invariant functions on $T^*\mathcal{O}(a)$. Then, according to the Noether theorem

$$\{\mathcal{F}_1^\epsilon, \mathcal{F}_2\}_\epsilon = 0,$$

where $\{\cdot,\cdot,\}_\epsilon$ are magnetic Poisson bracket with respect to $\omega + \epsilon\rho^*\Omega$.

Consider the Hamiltonian $H_\epsilon = \frac{1}{2}\langle \Phi_\epsilon, \Phi_\epsilon\rangle \in \mathcal{F}_1^\epsilon$. A simple calculation shows $H_\epsilon(x,p) = H_0 + \epsilon^2\frac{1}{2}\langle a, a\rangle$. Thus, we see that Hamiltonian flows of H_0 and H_ϵ coincides. Since H_ϵ belongs to $\mathcal{F}_1^\epsilon$ its commutes with $\mathcal{F}_2$. On the other side, as a composition of the momentum mapping with an invariant polynomial, the function H_ϵ is also G-invariant and commutes with $\mathcal{F}_1^\epsilon$. From the above consideration and Theorem 2.1 [3] we recover the Efimov result [6]:

Theorem 2 *Let G be a compact Lie group and $a \in \mathfrak{g}$. The magnetic geodesic flows of normal metric (9), (10) on the adjoint orbit $\mathcal{O}(a)$ is completely integrable in the non-commutative sense.*

Namely, the algebra of first integrals $\mathcal{F}_1^\epsilon + \mathcal{F}_2$ is complete on $(T^*\mathcal{O}(a), \omega + \epsilon\rho^*\Omega)$ (see [3]) and its invariant level sets are isotropic tori. Similarly as in the Liouville theorem, the tori are filled up with quasi-periodic trajectories of the system (9), (10) (see [9, 11]).

Integrable Deformations. Let $\mathcal{A} \subset \mathbb{R}(\mathfrak{g})$ be a commutative set of polynomials with respect to Lie-Poisson brackets on $\mathfrak{g}$. One can always find $\mathcal{A}$ that is complete on generic orbits $\mathcal{O}(\Phi_\epsilon(x,p))$ (e.g, see [2]). Let $\Phi_\epsilon^*\mathcal{A}$ be the pull-back of $\mathcal{A}$ by the momentum map: $\Phi_\epsilon^*\mathcal{A} = \{h \circ \Phi_\epsilon \,|\, h \in \mathcal{A}\}$.

Let $\mathcal{B}$ be a commutative subset of $\mathcal{F}_2$, with respect to the magnetic Poisson bracket. Then $\Phi_\epsilon^*\mathcal{A} + \mathcal{B}$ is a complete commutative set on $(T^*\mathcal{O}(a), \omega + \epsilon\rho^*\Omega)$ if $\mathcal{B}$ is a complete commutative subset of $\mathcal{F}_2$, i.e., we have

$$\delta = \dim \mathcal{O}(a) - \frac{1}{2}\dim \mathcal{O}(\Phi_\epsilon(x,p)) \qquad (17)$$

independent functions in $\mathcal{B}$, for a generic element $(x,p) \in T^*\mathcal{O}(a)$ [4].

The G-invariant, polynomial in momenta functions $f(x,p)$ on $T^*\mathcal{O}(a)$, are in one-to-one correspondence with Ad_{G_a}-invariant polynomials on $\mathfrak{v}$ via restriction to $T^*_a\mathcal{O}(a)$: $f_0(p_0) = f(a,p_0)$. Next, we apply the transformation

$$f_0 \mapsto \bar f, \quad f_0 = \bar f \circ \Phi_0|_{x=a} = \bar f \circ \mathrm{ad}_a.$$

Within these identifications, from (2), (4) and Thimm's formula for $\epsilon = 0$ [12], the magnetic Poisson bracket $\{f,g\}_\epsilon(x,p)$ corresponds to the following bracket (our notation is slightly different from Efimov's [6])

$$\{\bar f(\mu), \bar g(\mu)\}^\epsilon_{\mathfrak{v}} = -\langle \mu + \epsilon a, [\nabla \bar f(\mu), \nabla \bar g(\mu)]\rangle, \tag{18}$$

where $\mu = [a,p_0]$, $x = \mathrm{Ad}_g\, a$, $p = \mathrm{Ad}_g\, p_0$.

Note that $\{\{\cdot,\cdot\}^\lambda_{\mathfrak{v}},\ \lambda \in \mathbb{R}\}$ is a pencil of the compatible Poisson brackets on the algebra of Ad_{G_a}-invariant polynomials $\mathbb{R}[\mathfrak{v}]^{G_a}$. By the use of this pencil and the completeness criterion derived in [2], it is proved that the family of polynomials

$$\mathcal{B}_a = \{p^\lambda_a(\mu) = p(\mu + \lambda a),\ \lambda \in \mathbb{R},\ p \in \mathbb{R}[\mathfrak{g}]^G,\ \eta \in \mathfrak{v}\} \tag{19}$$

is a complete commutative subset of $\mathbb{R}[\mathfrak{v}]^{G_a}$ with respect to the canonical brackets $\{\cdot,\cdot\}^0_{\mathfrak{v}}$ (see [4, 7]). Here $\mathbb{R}[\mathfrak{g}]^G$ is the algebra of Ad_G-invariant polynomials on $\mathfrak{g}$. Using the method of [2], it can be verified that $\mathcal{B}_a$ is a complete commutative set with respect to the magnetic Poisson bracket (18) as well.

Let b an element from the center of $\mathrm{ann}(a)$. Define the *sectional operator* $\bar\phi_{a,b} : \mathfrak{v} \to \mathfrak{v}$ by $\bar\phi_{a,b} = \mathrm{ad}_a^{-1} \circ \mathrm{ad}_b = \mathrm{ad}_b \circ \mathrm{ad}_a^{-1}$. For compact groups, among sectional operators we can take positive definite ones. It easily follows from [8] that the function $\bar H_{a,b} = \frac{1}{2}\langle \bar\phi_{a,b}(\mu), \mu\rangle$ belongs to $\mathcal{B}_a$. The corresponding G-invariant function is

$$H_{a,b}(x,p) = \frac{1}{2}\langle \mathrm{ad}_{b_x} p, \mathrm{ad}_x p\rangle = -\frac{1}{2}\langle \mathrm{ad}_x \mathrm{ad}_{b_x} p, p\rangle = \frac{1}{2}\langle \phi_{x,b_x} p, p\rangle,$$

where $b_x = \mathrm{Ad}_g\, b$, $x = \mathrm{Ad}_g\, a$ and $\phi_{x,b_x} = -\mathrm{ad}_x \mathrm{ad}_{b_x}$. (Recall that b belongs to the center of $\mathrm{ann}(a)$ and since G is compact connected Lie group, G_a is also connected, so b_x is well defined.) This is a Hamiltonian function of the G-invariant metric $K_{a,b}$:

$$K_{a,b}(\eta_1,\eta_2) = \langle (\mathrm{ad}_{b_x})^{-1}\eta_1, \mathrm{ad}_x^{-1}\eta_2\rangle, \tag{20}$$

where $\eta_1, \eta_2 \in T_x\mathcal{O}(a)$. Whence, we get the following statement

Theorem 3 *The magnetic geodesic flows of the metrics $K_{a,b}$ with respect to the magnetic term $\epsilon\Omega$:*

$$\begin{aligned} \dot x &= -\mathrm{ad}_x \mathrm{ad}_{b_x} p = [[b_x, p], x], \\ \dot p &= -\mathrm{ad}_x^{-1}[p, [x, [b_x, p]]] + \mathrm{pr}_{\mathrm{ann}(x)}[[b_x, p], p] + \epsilon [b_x, p] \end{aligned}$$

are completely integrable in the commutative sense, by means of analytic, polynomial in momenta first integrals.

The Liouville Lagrangian tori are additionally foliated by δ-dimensional invariant isotropic tori, level sets of integrals $\mathcal{F}_1^\epsilon + \mathcal{B}_a$ (δ is given by (17)). Note that δ does not depend on ϵ: for a generic $\eta \in \mathfrak{v}$ we have equality $\dim \mathcal{O}(\eta + \epsilon a) = \dim \mathcal{O}(\eta)$ for all $\epsilon \in \mathbb{R}$ (see [4, 7]). Therefore, the influence of the magnetic fields $\epsilon\Omega$, $\epsilon \in \mathbb{R}$ reflects as a deformation of the foliation of the phase space $T^*\mathcal{O}(a)$ by invariant tori. As the magnetic field increases, the magnetic geodesic lines become more curved.

Concluding Remarks. One can take b such that the operator ϕ_{x,b_x} is positive, but with kernel different from zero. Then the Hamiltonian flow of $H_{a,b}$, for $\epsilon = 0$, represents an integrable *sub-Riemannian* geodesic flow on the orbit $\mathcal{O}(a)$ with the constraint distribution D at the point x given by the image

$$D_x = \phi_{x,b_x}(T_x^*\mathcal{O}(a)) = \mathrm{ad}_{b_x}(\mathrm{ann}(x))^\perp \subset T_x\mathcal{O}(a)$$

and the sub-Rieamannian structure defined by (20), where now $\eta_1, \eta_2 \in D_x$. Here we assume that the distribution D is bracket generating (see [10, 4] for more details).

There is a natural generalization of the above results to the class of magnetic potential systems on coadjoint orbits as well as to the wider class of homogeneous spaces. We shall consider these problems in the forthcoming paper.

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