Geodesic Flows: Chaos and Integrability

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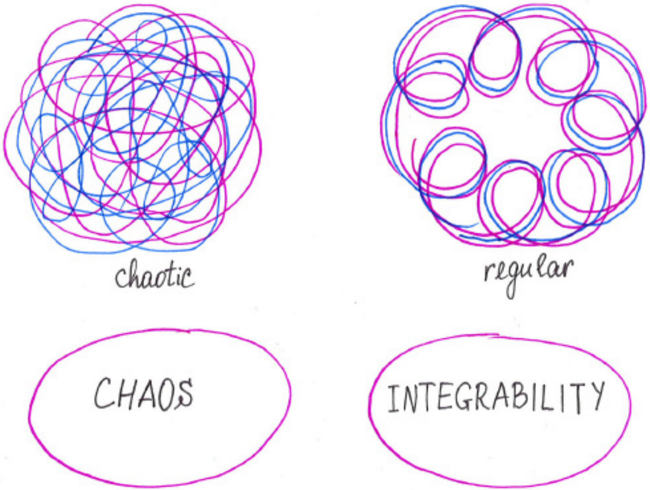
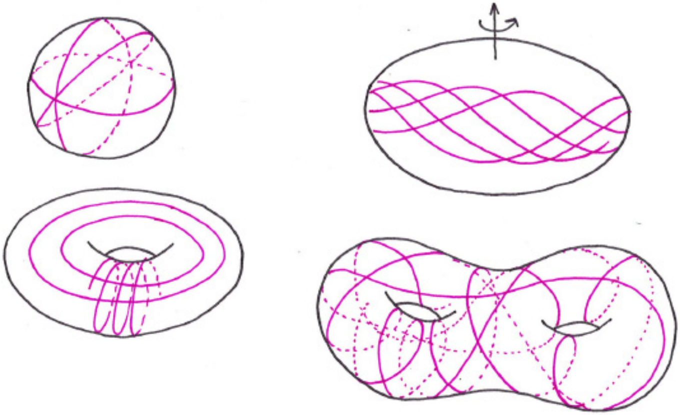
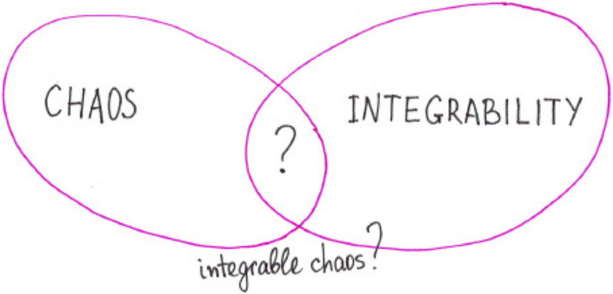
Chaos and Integrability



Examples of geodesic flows




CHAOS
INTEGRABILITY
?
integrable chaos?

What is Integrability? Liouville-Arnold Theorem

Hamiltonian systems:
(M^{2n}, ω) symplectic manifold,
$H : M \to \mathbb{R}$ smooth function (Hamiltonian)
Hamiltonian equations:

$$\frac{dx}{dt} = X_H(x) = \omega^{-1}(dH(x)) \quad \text{or, in coordinates:} \quad \begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q^i} \end{cases}$$

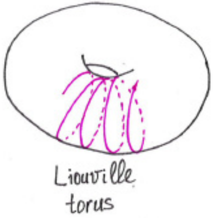
Complete integrability: There exist $F_1, \ldots, F_n$ which

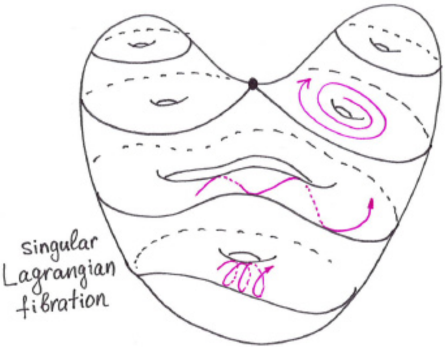
- are first integrals (i.e., preserved by the flow),
- pairwise commute $\{F_i, F_j\} = 0$,
- are functionally independent.

Theorem (Liouville-Arnold)

Let $X = \{F_1 = c_1, \ldots, F_n = c_n\}$ be regular, compact and connected. Then X is an n-dimensional torus and the dynamics on this torus is quasi-periodic.

Integrability

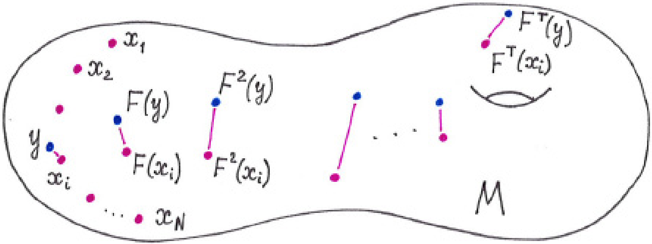




$F : M \to M$ a homeomorphism considered as a discrete dynamical system on a compact metric space (M, d): F^t, $t \in \mathbb{Z}$, $F^t = \underbrace{F \circ F \circ \cdots \circ F}_{t \text{ times}}$.

($F^t : M \to M$ the flow generated by a smooth vector field on a smooth compact manifold M)

Question: How much information do we need to approximate our dynamical system up to ϵ on the interval $t \in [0, T]$?



Let $N(\epsilon, T)$ be the minimal number of orbits sufficient for such an approximation: for any $y \in M$ there is x_i such that

$$d(F^t(y), F^t(x_i)) \leq \epsilon \text{ for any } t \in [0, T].$$

Formal definition and properties

Definition

The *topological entropy* of $F^t : M \to M$ is

$$h_{top}(F) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log N(\epsilon, T)}{T}$$

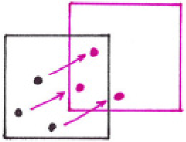
Properties:

- $h_{top}(F)$ does not depend on d, the only important thing is the topology on M.
- If $Y \subset X$ is a closed subspace invariant under F, then
$$h_{top}(F) \geq h_{top}(F|_Y).$$
- If $F : M \to M$ is an isometry (w.r.t. a certain metric d), then
$$h_{top}(F) = 0.$$

Two examples

Example: $F_1 : T^2 \to T^2$ translation on the 2-torus:

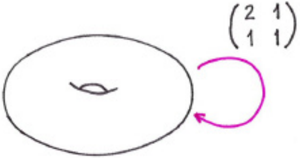
$$(x, y) \mod 2\pi \mapsto (x + a, y + b) \mod 2\pi$$

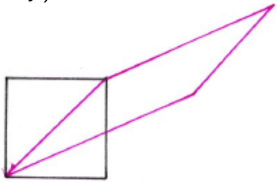


$$h_{top}(F_1) = 0.$$

Example: $F_2 : T^2 \to T^2$ Anosov hyperbolic automorphism (cat map):

$$(x, y) \mod 2\pi \mapsto (x + y, x + 2y) \mod 2\pi$$





$$h_{top}(F_2) = \log \frac{3 + \sqrt{5}}{2} > 0.$$

Problem: Describe all compact manifolds that admit integrable geodesic flows.
Inverse Problem: Describe topological obstructions to integrability.

Theorem (Kozlov, 1981, real analytic case)

In dim $= 2$*, integrable geodesic flows exist only on the sphere* S^2 *and torus* T^2*.*

Theorem (Dinaburg, 1971)

The topological entropy of any geodesic flow on M_g^2*,* $g > 1$*, is positive.*
If $\pi_1(M)$ *is of exponential growth then* $h_{top} > 0$ *for any geodesic flow.*

Theorem (Taimanov, 1987, geometrically simple case)

If M *admits an integrable geodesic flow, then* $\pi_1(M)$ *is almost abelian.*

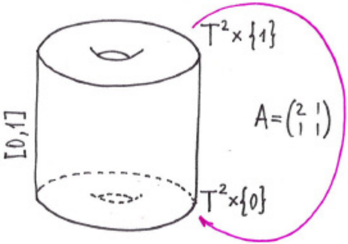
Theorem (Paternain, 1990)

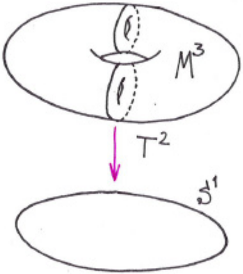
If a simply connected M *admits a geodesic flow with zero topological entropy, then* M *is rationally elliptic.*

Positive result: Integrable geodesic flows exists on all compact Lie groups, on their homogeneous spaces and bi-quotients (C^∞-case).

Example (L.Butler, I.Taimanov, A.B.)

Topology: M^3 is the three-dimensional closed manifold is obtained from $T^2 \times [0,1]$ by identifying $T^2 \times \{0\}$ with $T^2 \times \{1\}$ by means of the Anosov hyperbolic automorphism $F_A : T^2 \to T^2$, with $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. In other words, M^3 is a T^2-fiber bundle over S^1 with monodromy given by A.





Geometry: Riemannian metric on M^3 is:

$$ds^2 = ds_0^2(z) + dz^2, \text{ where } ds_0^2(z) \text{ is flat on } T^2\text{-fibers.}$$

For example:

$$ds^2 = \begin{pmatrix} dx & dy \end{pmatrix} A^{-2z} \begin{pmatrix} dx \\ dy \end{pmatrix} + dz^2$$

(M^3, g) is a standard SOL-manifold according to Thurston's classification.

Theorem

For (M^3, g):

- *geodesic flow is integrable by means of smooth integrals;*
- *its topological entropy is positive;*
- *$\pi_1(M)$ has exponential growth (in particular, is not almost abelian);*
- *geodesic flow is not integrable by means of real analytic integrals.*

Sketch of proof

We need three independent integrals. One of them is the Hamiltonian H. The coefficients of the metric depend only on z, but not on x and y. This implies that the momenta p_x and p_y are preserved by the geodesic flow, i.e., are its first integrals.

Problem: p_x and p_y are not well defined on M^3 (because of non-trivial monodromy).

Recipe: Replace p_x and p_y by functions $F_1(p_x, p_y)$ and $F_2(p_x, p_y)$ which are invariant under the monodromy map:

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \mapsto A^\top \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

Answer:

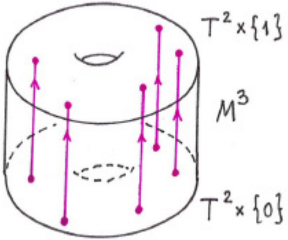
$$F_1 = p_x^2 - p_x p_y - p_y^2,$$

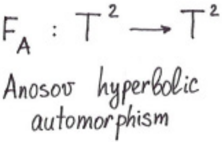
$$F_2 = \exp\left(-\frac{1}{F_1^2}\right) \cdot \sin\left(2\pi \frac{\log |p_x - \frac{1+\sqrt{5}}{2} p_y|}{\log \lambda}\right).$$

Singular set and entropy

Points with $p_x = p_y = 0$ form the singular set.
The geodesics lying in this set are very simple:

$$x(t) = \text{const}, \quad y(t) = \text{const}, \quad z(t) = t \qquad \text{vertical lines}$$





Conclusion: The time one map F^1 is exactly the Anosov automorphism $F_A : T^2 \to T^2$ with $h_{top}(F_A) > 0$.

Thus, the topological entropy of the geodesic flow on (M^3, g) ispositive.

Open questions

- Does analytic integrability imply $h_{top} = 0$?
- Does algebraic integrability imply $h_{top} = 0$?
- Obstructions to polynomial (algebraic) integrability.
- Obstructions to smooth integrability. Are there any geodesic flows on M_g^2, $g > 1$, integrable in C^∞ sense?
- If G/H is a homogeneous space of a compact Lie group, then the geodesic flow of the normal Riemannian metric on it admits a complete non-commutative algebra $\mathcal{F}$ of *polynomial* integrals. Does $\mathcal{F}$ always contain a complete commutative subalgebra?
- Relationship between "algebra" and "dynamics": what are the best left-invariant metrics on (non-compact) Lie groups which produce integrable geodesic flows?