Topology and Stability of Integrable Systems

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(joint work with A. Borisov and I. Mamaev)

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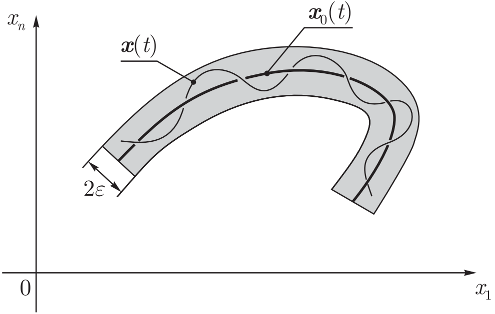
General problem:

Given a dynamical system $\dot{x} = v(x)$, we want to find all periodic stable trajectories.

Orbital stability:



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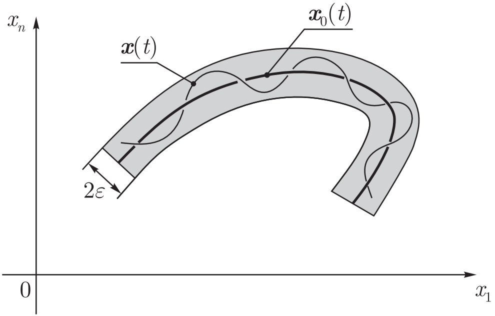

Specific question:
How to solve this problem in the case of integrable systems?

Assumption: Everything is real-analytic.

Integrability and Liouville fibration

Symplectic manifold $(\mathcal{M}, \omega)$

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$
Integrability: there exist $f_1, \ldots, f_n : \mathcal{M} \to \mathbb{R}$:

- first integrals of $X_H(x)$;
- commutative;
- independent almost everywhere.

For simplicity: two degrees of freedom

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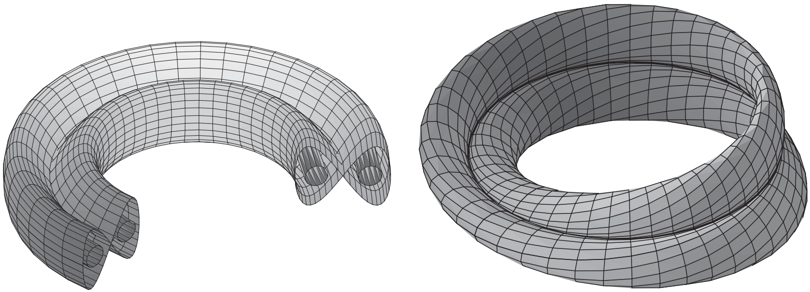
Bifurcation diagram $\Sigma = \{y \in \mathbb{R}^2 \mid y = \Phi(x),\ x \in S\}$

Illustration: Liouville fibration



Types of closed trajectories in integrable systems

Definition

A closed trajectory of an integrable Hamiltonian system is said to be *critical* if it is entirely contained in the set S of critical points of the integral map Φ otherwise it is said to be *non-critical*.

Since the critical set is invariant under a Hamiltonian flow, a non-critical trajectory γ does not intersect the critical set S.

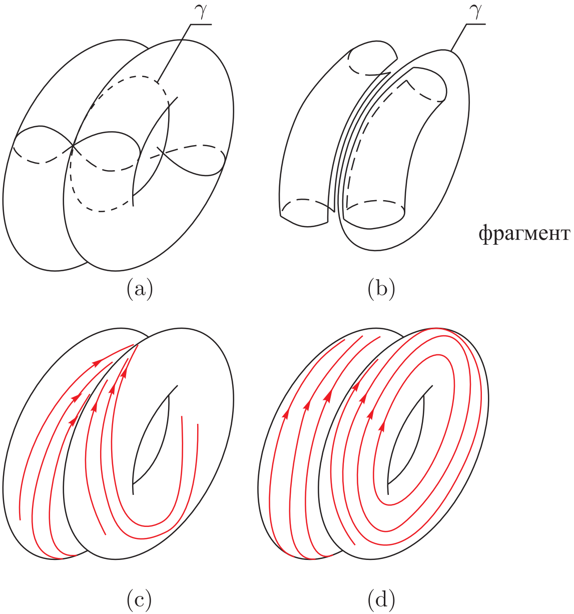
Critical closed trajectories can be of two types:

- non-degenerate,
- degenerate.

Non-critical closed trajectories can be of two types too:

- Closed trajectories that lie on invariant Liouville tori (these tori are automatically resonant, that is, consist entirely of closed trajectories)
- "Exceptional" closed trajectories that lie on singular integral manifolds $\{H = h_0, F = f_0\}$, but are not critical themselves.

Illustration: exceptional non-critical trajectories



Very important principle: Stable trajectories are always critical

Theorem

If the Hamiltonian system under consideration is non-resonant, then the non-critical closed trajectories are always unstable.

Proof.

If a non-critical periodic trajectory γ lies on a (resonant) Liouville torus, then its instability is quite obvious: near the given torus there is necessarily a non-resonant torus, on which the trajectories are dense; such trajectories cannot be contained entirely in a small neighborhood of γ.

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Comment.

The multipliers of non-critical trajectories are equal to 1. In this case, the stability analysis of a dynamical system by standard methods is very complicated. In our scheme, this does not lead to any additional difficulties. A good illustrating example could be the Goryachev–Chaplygin top.

Kolossoff solution.

Family of particular closed solutions of the Clebsch case that can be found explicitly in the form of quadratures.

These quadratures however are rather complicated and do not yield useful information about the qualitative properties of the solution.

Examples: Stability of Kolossoff solutions

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These quadratures however are rather complicated and do not yield useful information about the qualitative properties of the solution.

Description;

The principal moments of inertia are connected by the relation

$$I_3 = I_1 + I_2,$$

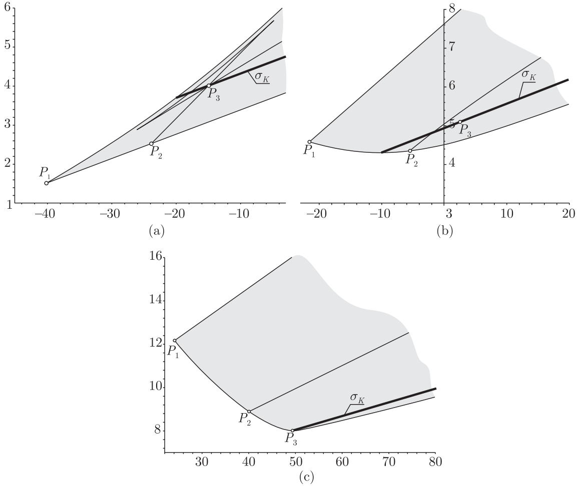
that is, the body is a flat plate. In this case, if the values of the Clebsch integral F and of the Hamiltonian H are connected by the linear relation

$$a_3 f - 2h + \frac{a_3^2 + a_1 a_2}{\det A} = 0, \qquad a_i = I_i^{-1}$$

then the system admits the additional rational integral

$$G = \frac{a_1 M_1 \gamma_1 + a_2 M_2 \gamma_2}{I_1 \gamma_2^2 - a_1 M_1^2}.$$

Kolossoff solutions on the bifurcation diagram



Conclusion: Kolossoff solutions are unstable (except, perhaps, for those of them which are critical).

Non-degenerate trajectories: two viewpoints

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Viewpoint of the theory of dynamical systems:
A periodic trajectory γ of a Hamiltonian system is *non-degenerate* if its multipliers are different from 1.

Due to conservation of the phase volume, the multipliers satisfy the relation $\lambda_1\lambda_2 = 1$. Depending on their values, two types of closed orbits are distinguished:

- *elliptic* if λ_1 and λ_2 are complex numbers lying on the unit circle,
- *hyperbolic* if λ_1 and λ_2 are real and with absolute values not equal to 1.

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Viewpoint of the theory of singularities:
A critical closed trajectory γ is *non-degenerate* if x_0 is a non-degenerate critical point for the function $F|_{\mathcal{N}_{x_0}}$ in the sense of Morse theory. In other words, the non-degeneracy of a trajectory in the sense of the theory of singularities means that it is non-degenerate as a critical submanifold of the function $F|_{\{H=\mathrm{const}\}}$

Furthermore, the trajectory is

- *elliptic* if the Hessian is sign-definite (that is, its eigenvalues are of the same sign),
- *hyperbolic* otherwise (the eigenvalues are of different signs).

Stability of non-degenerate trajectories and isolated families

Properties of non-degenerate critical trajectories:

- Every critical non-degenerate trajectory γ can be included in a one-parameter family γ_h of non-degenerate trajectories of the same type, and the value of the Hamiltonian H can be taken as a parameter.
- All together they form a two-dimensional invariant symplectic submanifold whose image under the integral map Φ is a smooth curve (one of the branches of the bifurcation diagram) which can be given on the plane $\mathbb{R}^2(h, f)$ as the graph of some smooth function $f = f(h)$.
- In a sufficiently small neighbourhood of the original trajectory $\gamma_0 = \gamma_{h_0}$ there are no other singular points of the integral map Φ (in other words, at least locally, the family $\{\gamma_h\}$ is isolated and does not interact with other critical trajectories or equilibrium points).

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Theorem (obvious and well known)

Hyperbolic trajectories are unstable, and elliptic trajectories are stable.

Isolated families of closed trajectories

Consider a one-parameter family of closed critical trajectories. Suppose that

- this family is isolated in the sense that there are no other critical points of the integral map Φ in a neighborhood of this family;
- this family is mapped onto some individual branch of the bifurcation diagram Σ which can be given as the graph of some smooth function $F = f(H)$.

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Theorem

Suppose that at least one trajectory in the family is non-degenerate. Then the following hold:

1. *almost all the trajectories of the family are non-degenerate and of the same type (either elliptic or hyperbolic);*
2. *if one of the trajectories of the family has elliptic type, then all the trajectories of the family (both degenerate and non-degenerate) are stable;*
3. *if at least one of the trajectories has hyperbolic type, then all the trajectories of the family (both degenerate and non-degenerate) are unstable.*

General Stability Criterion I

Theorem

A closed trajectory γ of an integrable Hamiltonian system is stable if and only if the connected component of the integral manifold containing this trajectory coincides with the trajectory itself, i.e.,

$$\gamma = \mathcal{M}_{h_0, f_0} = \{x \in \mathcal{M}^4 \mid H(x) = h_0, F(x) = f_0\},$$

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Proof. The proof in one direction, when a trajectory γ coincides with the integral submanifold, that is, has the form $\gamma = \{H = h_0, F = f_0\}$, is simple. We can take $(H - h_0)^2 + (F - f_0)^2$ as a Lyapunov function. The converse essentially uses the properties of singularities of real analytic functions.

Bifurcation Diagram and Bifurcation Complex

Take a regular point $(h, f) \in \Phi(\mathcal{M}) \setminus \Sigma$ in the image of the integral map Φ. Its preimage $\mathcal{M}_{h,f}$, i.e., the corresponding integral manifold, may contain more than one connected component (torus). We may think of this as different two-dimensional "leaves" over a regular region in $\Phi(\mathcal{M})$. Different leaves are glued together only along branches of the bifurcation diagram Σ.
Informally speaking, this collection of glued-together leaves and curves is called the bifurcation complex.

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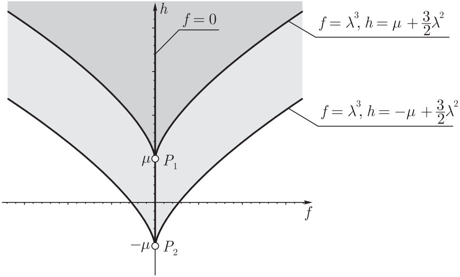
More rigorously:

Definition

The bifurcation complex $\mathcal{K}$ is the topological space whose points are defined to be the connected components of the integral manifolds $\mathcal{M}_{h,f}$ with the natural quotient topology.
Equivalently, $\mathcal{K}$ is the base of the Liouville fibration.

There are natural projection maps $\tilde{\Phi} : \mathcal{M} \to \mathcal{K}$ and $\pi : \mathcal{K} \to \Phi(\mathcal{M})$ such that $\Phi = \pi \circ \tilde{\Phi}$.

Illustration: Goryachev-Chaplygin case



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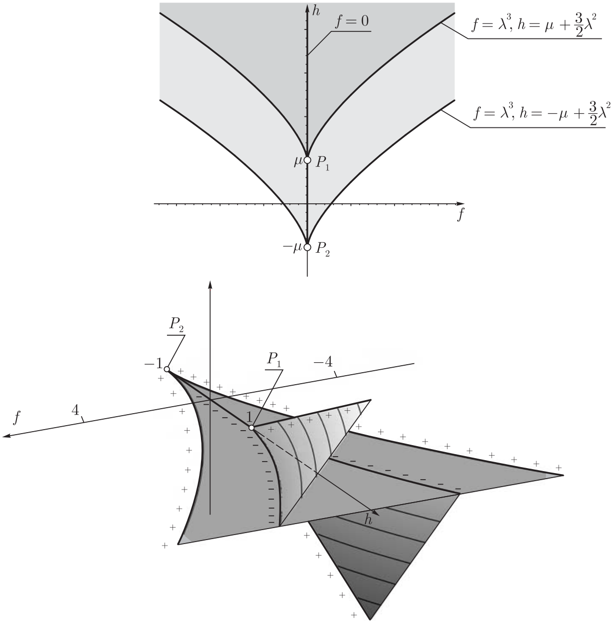

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Definition

We say that the momentum map Φ satisfies the *orbit finiteness condition* if each integral manifold $\mathcal{M}_{h,f}$ is a union of finitely many orbits of the Poisson action of $\mathbb{R}^2$ generated by the commuting functions H and F.

Equivalently: $\mathcal{M}_{h,f}$ contains only finitely many critical trajectories.

Example: Gaffet System

This system describes the evolution of relative sizes of an expanding gaseous ellipsoid filled with a monatomic ideal gas.

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Description:

The Hamiltonian in this case has the form

$$H = \frac{1}{2}M^2 + \frac{3}{2}\frac{a}{\gamma_1\gamma_2\gamma_3} + \frac{c^2}{(\gamma_1-\gamma_2)^2} + \frac{c^2}{(\gamma_1+\gamma_2)^2}$$

where the quantities γ_i are expressed in terms of the principle semi-axes A_i of the ellipsoid by the formulae $\gamma_i = A_i/\sqrt{\sum A_i^2}$. This system admits the integral of degree six

$$F_6 = (F_3 + F_c)^2 + 4\Phi\left(3a + G\frac{\gamma_1^2}{\gamma_3^2}\right)\left(3a + G\frac{\gamma_2^2}{\gamma_3^2}\right)$$

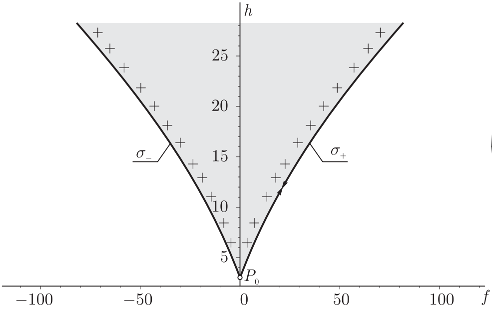
$$F_3 = M_1M_2M_3 - 3a(\gamma_1\gamma_2\gamma_3)^{1/3}\sum\frac{M_1}{\gamma_i}, \quad F_c = \frac{4c^2\gamma_1\gamma_2\gamma_3^2}{(\gamma_1^2-\gamma_2^2)^2}M_3,$$

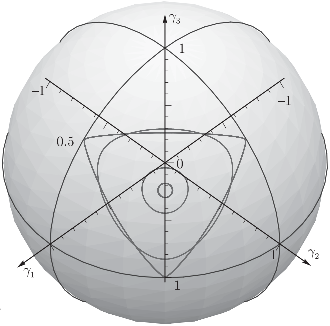
$$\Phi = 4c^2(\gamma_1\gamma_2\gamma_3)^{2/3}\frac{\gamma_3^2}{(\gamma_1^2-\gamma_2^2)^2}, \quad G = (\gamma_1\gamma_2\gamma_3)^{2/3}\frac{M_1M_2}{\gamma_1\gamma_2} + \Phi - 3a$$

For c = 0, we have the integral F_3 of degree three.

Example: Gaffet System

The bifurcation complex in this case consists of a single leaf.





Theorem

All the critical periodic solutions are stable, and there are no other stable solutions.