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Hamiltonization of Non-Holonomic Systems in the Neighborhood of Invariant Manifolds

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Abstract—The problem of Hamiltonization of non-holonomic systems, both integrable and non-integrable, is considered. This question is important in the qualitative analysis of such systems and it enables one to determine possible dynamical effects. The first part of the paper is devoted to representing integrable systems in a conformally Hamiltonian form. In the second part, the existence of a conformally Hamiltonian representation in a neighborhood of a periodic solution is proved for an arbitrary (including integrable) system preserving an invariant measure. Throughout the paper, general constructions are illustrated by examples in non-holonomic mechanics.

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Contents

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INTRODUCTION

The present paper is devoted to the problem of Hamiltonization of non-holonomic systems. It is desirable, in the most general setting, to try to find obstructions to Hamiltonization or, on the contrary, to prove that a Hamiltonian representation does exist. If a system as a whole is considered, then questions of this kind are rather difficult. However, the Hamiltonization problem can be naturally posed in a more local context, that is, in a neighborhood of an equilibrium, a closed trajectory, or an integral manifold.

In this paper we prove several results on this subject and discuss a number of examples.

Definition. *A dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ on a manifold $\mathcal{M}$ is said to be conformally Hamiltonian if there exists a Poisson structure $\mathbf{J}(\boldsymbol{x})$ on $\mathcal{M}$ such that $\boldsymbol{v} = \lambda(\boldsymbol{x})\mathbf{J}\,dH$ for some suitable function $H(\boldsymbol{x})$ and an everywhere positive function $\lambda(\boldsymbol{x})$.*

First of all we point out that a Hamiltonian (and, of course, a conformally Hamiltonian) system must have a first integral and an invariant measure. These conditions are necessary for Hamiltonization. Therefore we assume them here. We also assume everywhere that the invariant measure is smooth and its density does not vanish anywhere. We identify the measure with the differential form $\mu = \rho(\boldsymbol{x})\,dx_1 \wedge \cdots \wedge dx_n$ of maximal rank.

Finally we point out two more simple but important circumstances related to invariant measures.

Proposition 1. *Suppose that a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ on a manifold $\mathcal{M}^n$ has an invariant measure μ and first integrals $f_1, \ldots, f_k$. Suppose that an integral surface*

$$\mathcal{X} = \{f_1 = c_1, \ldots, f_k = c_k\}$$

is regular, that is, $df_1(\boldsymbol{x}), \ldots, df_k(\boldsymbol{x})$ are linearly independent at all points $\boldsymbol{x} \in \mathcal{X}$. Then the restriction of the dynamical system to $\mathcal{X}$ also has an invariant measure that is defined by a $(n-k)$-form σ satisfying the identity

$$\sigma \wedge df_1 \wedge \cdots \wedge df_k = \mu.$$

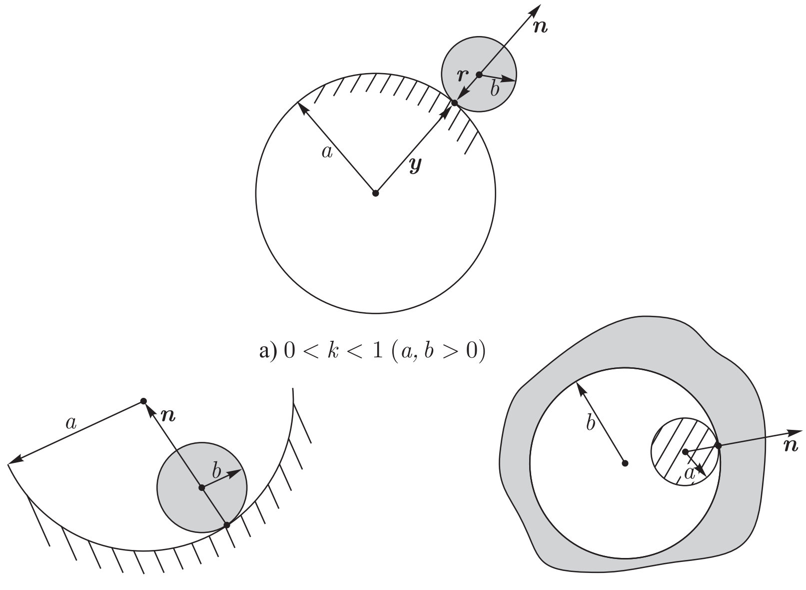
Proposition 2. *Suppose that a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ on a manifold $\mathcal{M}^n$ has an invariant measure μ and suppose that a (generally speaking, local) Poincaré section $\mathcal{N}^{n-1}$ transversal to the flow and the Poincaré map $\Phi\colon \mathcal{N}^{n-1} \to \mathcal{N}^{n-1}$ can be defined for it. Then the Poincaré map also has an invariant measure on $\mathcal{N}^{n-1}$ and this measure is given by the $(n-1)$-form $\zeta(\cdot) = \mu(\boldsymbol{v}, \cdot)$.*

We shall illustrate the theoretical results presented below by the examples of two systems of non-holonomic mechanics:

- the first system describes the rolling without slipping of an inhomogeneous balanced body with a spherical shell (that is, of a dynamically asymmetric ball whose center of mass coincides with the geometric center) on a spherical base; we call it for brevity the *Chaplygin ball on a sphere*;

- the second system also describes the rolling of an inhomogeneous balanced body with a spherical shell on a spherical base but under the condition of absence of both slipping and twisting (that is, rotation about the normal at the contact point); we call this system the *rubber Chaplygin ball on a sphere.*

Remark. In [1] possible realizations are discussed of the non-holonomic constraint consisting in the absence of twisting that are not necessarily associated with a rubber shell of the ball. The above-mentioned name is fairly relative, it is applied for a brief designation of this class of systems and was introduced in [2].



a) $0 < k < 1$ $(a, b > 0)$

b) $k > 1$ $(a > 0,\ b > 0,\ |b| < a)$

c) $k < 0$ $(a < 0,\ b > 0,\ |a| < b)$

Fig. 1. Scheme of the rolling of a ball (denoted by grey) on a stationary spherical base (denoted by hatching) for various signs of the curvature radia a, b.

We now present the equations of motion of these systems and indicate their invariant measure and most general first integrals existing for arbitrary values of the parameters.

Chaplygin ball on a sphere. This problem was considered from various positions in [3] and is a generalization of the classical Chaplygin problem about the rolling of a ball on a plane [4]. Suppose that a dynamically asymmetric balanced rigid body with a spherical surface (the Chaplygin ball) rolls without slipping on the surface of a stationary sphere of radius a; depending on the signs of the curvature, there are three possible variants of rolling (see Fig. 1). We choose the moving coordinate system attached to the principal axes of the ball, and denote by $\boldsymbol{\omega}$, m, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$, and b the angular velocity, mass, inertia tensor, and radius of the ball, respectively. The angular momentum of the ball with respect to the tangent point Q has the form

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + D\boldsymbol{n} \times (\boldsymbol{\omega} \times \boldsymbol{n}), \qquad D = mb^2, \tag{1}$$

where $\boldsymbol{n}$ is the normal vector to the sphere at the contact point (see Fig. 1). The full phase space of the system is $T(SO(3) \times S^2)$. We postulate that the potential energy of the ball depends only on the position of its center; then, by using the non-holonomic constraint — the absence of slipping (that is, the velocity of the contact point is equal to zero), we can obtain the equations of motion of the reduced system in the following closed form:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + k\boldsymbol{n} \times \frac{\partial U}{\partial \boldsymbol{n}}, \qquad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega}, \qquad k = \frac{a}{a+b}, \tag{2}$$

where U is the potential of external forces, a is the radius of the stationary sphere (note that the radia of the moving and stationary spheres must be taken with the appropriate sign, see Fig. 1), and the angular velocity is expressed in terms of the angular momentum by using relation (1) by the formula

$$\boldsymbol{\omega} = \mathbf{A}\boldsymbol{M} + \Lambda\mathbf{A}\boldsymbol{n},$$
$$\mathbf{A} = (\mathbf{I} + D\mathbf{E})^{-1}, \quad \Lambda = \frac{(\mathbf{A}\boldsymbol{M}, \boldsymbol{n})}{(\boldsymbol{n}, \boldsymbol{n})D^{-1} - (\boldsymbol{n}, \mathbf{A}\boldsymbol{n})}.$$

The system (2) has the invariant measure $\rho\, d\boldsymbol{M}\, d\boldsymbol{n}$ with the density

$$\rho = \big((\boldsymbol{n}, \boldsymbol{n})D^{-1} - (\boldsymbol{n}, \mathbf{A}\boldsymbol{n})\big)^{-1/2}$$

and admits two (general) first integrals — the geometric one and the energy:

$$F_0 = (\boldsymbol{n}, \boldsymbol{n}) = 1,$$
$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{n}) = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + \frac{1}{2}D\Lambda(\mathbf{A}\boldsymbol{M}, \boldsymbol{n}) + U(\boldsymbol{n}). \tag{3}$$

For $U \equiv 0$ and an arbitrary k there is one more integral

$$F_1 = (\boldsymbol{M}, \boldsymbol{M}). \tag{4}$$

Rubber ball on a sphere. This problem was considered for the first time in [5] and was studied in more detail in [2, 6]. As above, we consider the motion of a dynamically asymmetric balanced ball on the surface of a stationary sphere, using the same notation for the corresponding quantities as above (see Fig. 1). We assume that during the motion the instantaneous velocity of the contact point and the projection of the angular velocity of the ball onto the normal to the sphere are equal to zero. This model of motion differs from the classical non-holonomic model of rolling without slipping in which it is only assumed that the velocity of the contact point is equal to zero.

As above, we choose the moving coordinate system connected with the principal axes of the ball. Here, the kinematic equations describing the evolution of the normal $\boldsymbol{n}$ coincide with the previous ones (2), while it is more convenient to write the dynamical equations in terms of the angular velocity $\boldsymbol{\omega}$. Finally, for the motion in a potential force field $U(\boldsymbol{n})$ depending only on the normal, we obtain

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \lambda\boldsymbol{n} + k\boldsymbol{n} \times \frac{\partial U}{\partial \boldsymbol{n}}, \quad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega},$$
$$\mathbf{J} = \mathbf{I} + mb^2\mathbf{E}, \quad \mathbf{E} = \|\delta_{ij}\|, \tag{5}$$

where

$$\lambda = -\frac{(\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega}, \mathbf{J}^{-1}\boldsymbol{n}) + \left(k\boldsymbol{n} \times \dfrac{\partial U}{\partial \boldsymbol{n}}, \mathbf{J}\boldsymbol{n}\right)}{(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}.$$

The system (5) has the invariant measure

$$(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{\frac{1}{2k}}\, d\boldsymbol{\omega}\, d\boldsymbol{n}. \tag{6}$$

The measure (6) was found in [2].

The constraint

$$F_1 = (\boldsymbol{\omega}, \boldsymbol{n}) = 0 \tag{7}$$

can also be regarded as an additional partial integral. Furthermore, equations (5) also have the energy and geometric integrals:

$$H = \frac{1}{2}(\mathbf{J}\boldsymbol{\omega}, \boldsymbol{\omega}) + U(\boldsymbol{n}), \quad F_0 = (\boldsymbol{n}, \boldsymbol{n}) = 1. \tag{8}$$

1. HAMILTONIZATION IN A NEIGHBORHOOD OF A TWO-DIMENSIONAL INVARIANT MANIFOLD

1.1. Theorem on Rectification of a Vector Field

First of all we discuss a theorem that is, to some extent, an analogue of the Liouville–Arnol'd theorem for Hamiltonian systems. This theorem is most often used for establishing the integrability of systems that have an invariant measure (in particular, non-holonomic systems).

The first versions of this assertion, which make it possible to integrate equations of motion by quadratures, were used as back as in the works of Euler and Jacobi; therefore in some papers it is called the Euler–Jacobi theorem. More precisely, in the XIXth century this method (under the name *last multiplier theory*) was also used for integration in dynamics of a rigid body (the Euler–Poisson, Kirchhoff equations), where the Hamiltonian structure of equations was established later [7, 8].

The statement presented here is a natural generalization of the theorem stated by V. V. Kozlov [9] and called the Euler–Jacobi theorem. The difference is in the fact that here reduction of the flow to the standard form is considered not on an individual torus (where A. N. Kolmogorov's theorem [10] can be applied) but on an entire analytic family of tori. This generalization is possible due to the fact that we in addition require the analyticity of the dynamical system itself, its first integrals and invariant measure.

Here we present a complete proof of this theorem, in view of the fact that the proof of such an assertion, although seemingly obvious, has not been given anywhere with sufficient degree of rigor and completeness.

Theorem 1. *Suppose that on an n-dimensional orientable manifold we are given a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ that has independent integrals $f_1, \ldots, f_{n-1}$ and a smooth invariant measure μ.*[1)] *Consider a nonsingular integral surface*

$$\mathcal{X} = \{f_1 = c_1, \ldots, f_{n-2} = c_{n-2}\}.$$

Suppose that $\mathcal{X}$ is compact and the vector field $\boldsymbol{v}$ does not vanish on $\mathcal{X}$. Then $\mathcal{X}$ and all the nearby integral surfaces are two-dimensional tori. Furthermore, in some neighborhood of $\mathcal{X}$ there exist local coordinates $\phi_1, \phi_2, f_1, \ldots, f_{n-2}$ such that ϕ_1, ϕ_2 are angular coordinates on the tori and the dynamical system takes the form

$$\dot{f}_i = 0, \quad \dot{\phi}_i = \frac{\lambda_i}{\Phi},$$

where $\lambda = \lambda_i(f_1, \ldots, f_{n-2})$ and $\Phi = \Phi(\phi_1, \phi_2, f_1, \ldots, f_{n-2})$.

If the original dynamical system, its integrals and invariant measure were real analytic, then so are also the functions λ_i, Φ and the angular variables ϕ_1, ϕ_2.

Proof. For the proof, we show that the vector field $\boldsymbol{v}$, suitably normalized, admits an independent symmetry field $\boldsymbol{u}$ (that is, $[\Phi(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{u}] = 0$ for some function $\Phi(\boldsymbol{x})$) and then use the standard assertion that for two commuting vector fields on a torus there exist angular variables in which these fields are rectified.

We divide the proof into several steps.

First of all we point out that $\mathcal{X}$ is a torus due to the orientability and existence on it of a vector field without singular points. Next, the smooth measure μ can be naturally restricted to the torus $\mathcal{X}$ and regarded as a symplectic structure $\boldsymbol{\omega}$ on this torus. Therefore the condition of preservation of the measure is equivalent to the local Hamiltonianity of the flow of $\boldsymbol{v}$ on $\mathcal{X}$. In particular, for the vector field $\boldsymbol{v}$ there exists a multi-valued Hamiltonian H without singular points.[2)] The differential of the Hamiltonian is single-valued; being a closed form on the torus, this differential can always be written in the form[3)]

$$dH = c_1\,dx_1 + c_2\,dx_2 + df(x_1, x_2),$$

where x_1, x_2 are angular coordinates on the torus with period 1, while $f(x_1, x_2)$ is some periodic function (that is, a single-valued function on the torus). We could say equivalently that the multi-valued Hamiltonian has the form $H = c_1x_1 + c_2x_2 + f(x_1, x_2)$. The constants c_1 and c_2 can be determined by integration: $c_i = \int_{\gamma_i} dH$, where γ_i is the basis cycle corresponding to the angular coordinate x_i.

Consider an arbitrary Riemannian metric ds^2 on the torus (for example, $dx_1^2 + dx_2^2$) and the vector field $\boldsymbol{\xi} = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|^2}$. This vector field is well defined and has the remarkable property that its flow preserves the trajectories of the vector field $\boldsymbol{v}$. Moreover, if we bear in mind that the trajectories are the level lines of the 'multi-valued Hamiltonian', then under the translation along

[1)] It is possible not to assume the orientability but say instead that the measure is given by a differential n-form.

[2)] One should not look for an analogy with the standard situation where the Hamiltonian is a constant on invariant tori. Here H plays an auxiliary role and has completely different nature.

[3)] Of course, the form dH can be written explicitly in terms of $\boldsymbol{v}$ and $\boldsymbol{\omega}$. Namely, $dH(\cdot) = \boldsymbol{\omega}(\boldsymbol{v}, \cdot)$.

the flow of $\boldsymbol{\xi}$ by time $t = c$ the level $\{H = h\}$ goes exactly to the level $\{H = h + c\}$. This immediately implies that all the trajectories are either simultaneously closed, or simultaneously non-closed.

In essence, $\boldsymbol{\xi}$ is already almost a symmetry field, but for the moment it preserves not the vector field $\boldsymbol{v}$ itself but only its trajectories. For constructing a genuine symmetry field we shall need to modify $\boldsymbol{\xi}$, making all its trajectories closed. First we make one trajectory closed. For that we consider the vector field $\widetilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \lambda \boldsymbol{v}$, $\lambda \in \mathbb{R}$. The following assertion is in fact quite obvious and is one of the versions of the well-known Siegel lemma [11].

Lemma 1. *There exists $\lambda \in \mathbb{R}$ such that at least one of the trajectories of the vector field $\widetilde{\boldsymbol{\xi}}$ is closed, transversal to the vector field $\boldsymbol{v}$, and realizes a homotopically nontrivial cycle α on the torus*[4].

Proof. Consider an arbitrary point $P_0 \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the trajectory γ_0 of the vector field $\boldsymbol{v}$ passing through P_0. We lift this trajectory to the covering plane $\mathbb{R}^2$, where it is a level line of the Hamiltonian: $\gamma_0 = \{H = 0\}$ for definiteness.

If γ_0 is non-closed, then we choose an arbitrary homotopically nontrivial cycle α on the torus; if γ_0 is closed and itself realizes some nontrivial cycle β, then we choose as α an arbitrary cycle complementary to β. On the covering plane $\mathbb{R}^2$, to the cycle α there corresponds the translation by the corresponding element of the lattice. Let $P_0' \in \mathbb{R}^2$ be the point obtained from P_0 by this translation. Consider the trajectory of the vector field $\boldsymbol{v}$ passing through P_0'. Of course, these two trajectories coincide on the torus, but they are distinct on the covering plane, so that this trajectory can be represented as $\gamma_c = \{H = c\}$, $c \neq 0$ (Fig. 2).

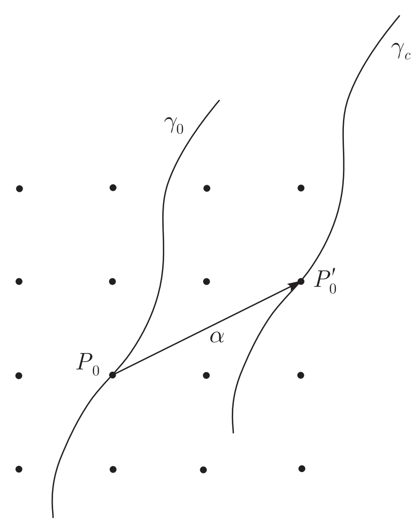


Fig. 2. Trajectories of the vector field $\boldsymbol{v}$ on the covering plane.

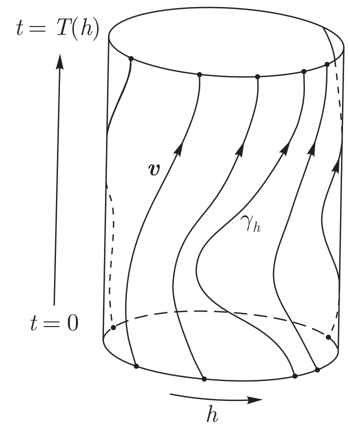


Fig. 3. Poincaré section of the flow of the vector field $\boldsymbol{v}$ by a closed trajectory of the field $\widetilde{\boldsymbol{\xi}}$.

Consider the flow of the vector field $\widetilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \lambda \boldsymbol{v}$ on the covering plane $\mathbb{R}^2$. This flow preserves the trajectories of $\boldsymbol{v}$ and, moreover, the shift by time $t = c$ takes γ_0 to γ_c. Then the point P_0 goes to some point $P' \in \gamma_c$. For the trajectory of the vector field $\widetilde{\boldsymbol{\xi}}$ to become closed on the torus, it is necessary and sufficient that P' coincides with P_0'. Clearly, as λ varies, the point P' continuously moves along the trajectory γ_c from $-\infty$ to $+\infty$. Therefore there exists a value λ such that P' coincides with P_0', as required. The lemma is proved.

[4] Moreover, if the trajectories of $\boldsymbol{v}$ are non-closed, then this cycle α can be made absolutely arbitrary by a choice of λ. If the trajectories of $\boldsymbol{v}$ are closed and themselves realize some nontrivial cycle β, then we can obtain as α any cycle that is non-homotopic to β.

The next step in the proof is to modify $\widetilde{\boldsymbol{\xi}}$ once again in such a way that already all its trajectories become closed. We do this in two stages. First we show that this can be done in the sense of C^∞.

We regard the closed trajectory of the vector field $\widetilde{\boldsymbol{\xi}}$ constructed above as a global Poincaré section for the flow of $\boldsymbol{v}$. If we cut the torus along this trajectory, then the torus turns into a cylinder fibred into trajectories of the vector field $\boldsymbol{v}$ going from the bottom base of the cylinder to the top one (Fig. 3). As natural coordinates on the cylinder we can introduce h and t, where $h \in [0, c]$ is the value of the Hamiltonian parameterizing the trajectories of the vector field $\boldsymbol{v}$, and t is the time along these trajectories counted from the bottom base of the cylinder. We point out that on every individual trajectory $\gamma_h = \{H = h\}$ the parameter t varies from zero to some quantity $T(h)$, which can be interpreted as the length of the part of the trajectory enclosed between the bottom and top bases of the cylinder or, which is the same, the time spent by a point to perform one revolution and return to the Poincaré section. The quantity $T(h)$ is not constant but is a periodic function of $h \in [0, c]$.

We wish to "re-parameterize the system" in such a way that the time of one revolution be the same and equal to 1 for all points. We define a new parameter $\tau = \tau(t, h)$ by the relation

$$t(\tau) = \int_0^\tau (1 + s(h)\psi(t))\, dt,$$

where $\psi(t)$ is a standard smoothing function, the graph of which is depicted in Fig. 4, and $s(h)$ is chosen so that for the trajectory $\gamma_h = \{H = h\}$ we have the relation

$$t(1) = \int_0^1 (1 + s(h)\psi(t))\, dt = T(h), \qquad \text{that is,} \quad s(h) = \frac{T(h) - 1}{\int_0^1 \psi(t)\, dt}.$$

If ε is sufficiently small (see Fig. 4), then $s(h) > -1$. Therefore the integrand is strictly positive and the change of variable is monotonic.

The smoothing is necessary for the re-parametrization to be C^∞-smooth after gluing together the top and bottom bases of the cylinder. This choice of the function ensures that $d\tau/dt \equiv 1$ in a neighborhood of the bases being glued together, and therefore, after the inverse identification of the bases of the cylinder, the function τ can now be interpreted as a new angular coordinate on our torus (everywhere independent of H!).

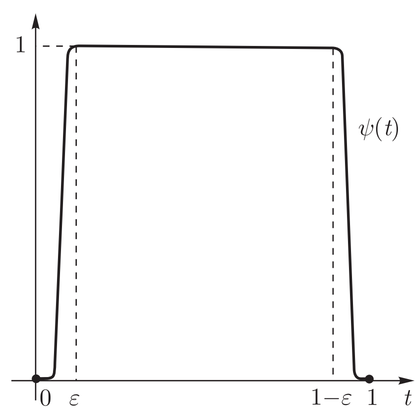


Fig. 4. Smoothing function.

By construction, τ is C^∞-smooth but not real-analytic. However, we can choose a new analytic angular coordinate $\widetilde{\tau}$ so that it is arbitrarily close to τ in the C^2-topology (see Appendix), which ensures that it is independent of H (no other properties of $\widetilde{\tau}$ will be needed).

Thus, we can assume without loss of generality that the angular coordinate τ is real-analytic. Clearly, τ can be analytically extended to all the nearby tori. The Hamiltonian H is also defined on all the nearby tori and, of course, depends analytically on the first integrals $f_1, \cdots, f_n$. The independence of H and τ on nearby tori is preserved.

In what follows we can interpret τ and H as two 'local coordinates' on the tori and consider the coordinate vector fields ∂_τ and ∂_H. Note that ∂_τ is tangent to the level lines $\{H = h\}$, that is, the trajectories of the vector field $\boldsymbol{v}$, but because of the re-parametrization conducted above, ∂_τ differs from the vector field $\boldsymbol{v}$ itself by some positive factor, that is, $\partial_\tau = \Phi \cdot \boldsymbol{v}$. The trajectories of the field $\boldsymbol{u} = \partial_H$ are closed and homotopic to the cycle α. Being coordinate vector fields, ∂_τ and ∂_H commute, that is, $\boldsymbol{u} = \partial_H$ is a symmetry field for $\Phi \cdot \boldsymbol{v}$, as required.

It remains to use the following standard fact: if two commuting vector fields are given on a torus, then there exists a system of angular coordinates such that both fields are rectified (see Appendix). The proof of this fact is simple: two commuting vector fields define an action of the group $\mathbb{R}^2$ on the torus. Clearly, the torus is an orbit of this action and therefore it can be represented as the quotient of $\mathbb{R}^2$ by some lattice. Choosing a basis of the lattice is equivalent to choosing angular coordinates. The theorem is proved. □

Comment. There is an interesting question of rectification of a vector field on tori without change of time [9]. This question turns out to be quite nontrivial (even on an individual torus) and is related to the properties of the rotation number (Kolmogorov [10]). If the torus is resonant, that is, all the trajectories on it are closed, then a necessary and sufficient condition of rectification is that their periods be equal. In non-holonomic systems, the periods of trajectories on resonant tori do not have to be equal; therefore rectification (and therefore Hamiltonianity) in general cannot be achieved without re-parametrization; see [12]. (In that paper, resonant tori in the problem about the Chaplygin ball were considered as an example, and a difference in the periods for trajectories on one and the same torus was established numerically.) Kolmogorov's theorem asserts that rectification of the field without change of time is nevertheless possible for almost all irrational rotation numbers and gives the corresponding characterization of these numbers (they must satisfy a certain Diophantine condition; see also [13]).

Remark. There is also a simplest special case noted in [14] when rectification of the vector field without change of time is possible on a family of tori. Here, the vector field must satisfy a certain additional requirement. Such systems were called Lagrange–Lie integrable in [14].

1.2. Hamiltonization in a Neighborhood of a Torus

The theorem proved above asserts that the dynamics of a system having $(n-2)$ integrals and an invariant measure, after a suitable change of time, becomes in fact the same as for Liouville integrable Hamiltonian systems. Furthermore, it turns out that the Poisson structure and the Hamiltonian can be indicated not uniquely. In fact, the following assertion holds.

Under the hypothesis of the preceding theorem, we distinguish one of the integrals which we wish to see as the Hamiltonian of the system; among the remaining integrals we arbitrarily choose $n-4$ functions that will play the role of the Casimir functions. We fix this choice by denoting the available $n-2$ integrals of the vector field $\boldsymbol{v}$ by $H, f, C_1, \ldots, C_{n-4}$. Then our vector field is conformally Hamiltonian in the following precise sense.

Theorem 2. *Suppose that the hypotheses of the preceding theorem hold. Then in a neighborhood of the integral surface $\mathcal{X}$ there exists a Poisson structure $\mathbf{J}$ such that $C_1, \ldots, C_{n-4}$ are its Casimir functions, and $\boldsymbol{v} = \lambda \mathbf{J}\, dH$, where λ is some smooth positive function. In the real-analytic case, the Poisson structure $\mathbf{J}$ and the factor λ are also real-analytic.*

Remark. We point out that the function λ is, generally speaking, different from the factor Φ^{-1} in the preceding theorem, and the Hamiltonian H does not coincide with the multi-valued Hamiltonian defined in the proof of the preceding theorem.

Proof. Of course we shall be using the preceding theorem. We denote by $\rho = \dfrac{\lambda_1}{\lambda_2}$ the rotation number of the vector field $\boldsymbol{v}$. For definiteness we assume that $\lambda_2 \neq 0$ (otherwise we can interchange λ_1 and λ_2).

Instead of seeking an explicit formula for the Poisson structure $\mathbf{J}$ in the coordinates ϕ_1, ϕ_2, H, f, $C_1, \dots, C_{n-4}$, we seek canonical coordinates for this Poisson structure of the type of action-angle variables. For that we define two functions

$$I_1 = I_1(H, f, C_1, \dots, C_{n-4}) \quad \text{and} \quad I_2 = I_2(H, f, C_1, \dots, C_{n-4})$$

such that our vector field $\boldsymbol{v}$ is proportional to the Hamiltonian vector field $\mathbf{J}\, dH$, where

$$\mathbf{J} = \partial_{\phi_1} \wedge \partial_{I_1} + \partial_{\phi_2} \wedge \partial_{I_2}. \tag{9}$$

As I_1 we choose an absolutely arbitrary function that is independent of H and the Casimir functions $C_1, \dots, C_{n-4}$. Then the rotation function ρ can be written as a smooth (analytic) function of $I_1, H, C_1, \dots, C_{n-4}$.

We define the second 'action variable' by the following explicit formula:

$$I_2 = -\int \rho(I_1, H, C_1, \dots, C_{n-4})\, dI_1.$$

(In other words, I_2 is determined from the equation $\dfrac{\partial I_2}{\partial I_1} = -\rho$.) This function is determined up to addition of an arbitrary function of $H, C_1, \dots, C_{n-4}$. In particular, we can assume that $\dfrac{\partial I_2}{\partial H} \neq 0$ (the partial differentiation is performed in the local coordinates I_1, H, $C_1, \dots, C_{n-4}$). This condition guarantees that $I_1, I_2, C_1, \dots, C_{n-4}$ are independent, and therefore formula (9) makes sense and defines some Poisson structure. We now show that this structure is the sought-for one.

Let us find the vector field $\mathbf{J}\, dH$. First, we point out that on every torus this Hamiltonian vector field is rectified in the variables ϕ_1, ϕ_2 (in the same fashion as the vector field $\boldsymbol{v}$). This follows immediately from the fact that $H = H(I_1, I_2, C_1, \dots, C_{n-4})$ and $\mathbf{J}$ has canonical form. Therefore, in order to show that $\boldsymbol{v}$ and $\mathbf{J}\, dH$ are proportional, it is sufficient to verify that the rotation numbers of these vector fields coincide.

This can be easily verified by a straightforward calculation. Indeed,

$$\mathbf{J}\, dH = \frac{\partial H}{\partial I_1}\partial_{\phi_1} + \frac{\partial H}{\partial I_2}\partial_{\phi_2};$$

hence,

$$\rho_H = \frac{\frac{\partial H}{\partial I_1}}{\frac{\partial H}{\partial I_2}}.$$

Consider the following regular change of variables (here the Casimir functions are regarded as parameters that are not involved in the change of variables):

$$I_1, H \to I_1, I_2;$$

The Jacobi matrix of this change of variables has the form

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial H}{\partial I_1} & \frac{\partial H}{\partial I_2} \end{pmatrix}.$$

The matrix of the inverse change of variables has the form

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial I_2}{\partial I_1} & \frac{\partial I_2}{\partial H} \end{pmatrix}.$$

Since these two matrices are inverse to each other, we obtain

$$\frac{\partial H}{\partial I_1} + \frac{\partial H}{\partial I_2}\frac{\partial I_2}{\partial I_1} = 0.$$

Hence,

$$\rho_H = \frac{\frac{\partial H}{\partial I_1}}{\frac{\partial H}{\partial I_2}} = -\frac{\partial I_2}{\partial I_1} = \rho,$$

as required. The theorem is proved. □

1.3. Examples

We illustrate the theorem proved above by examples of integrable non-holonomic systems in which a conformally Hamiltonian representation was found.

Example 1 (Chaplygin ball on a plane). Consider the system (2) for $k = 1$ and $U = 0$, which describes the rolling without slipping of a dynamically asymmetric balanced ball on a stationary horizontal plane. In this case, the area integral

$$F_2 = (\boldsymbol{M}, \boldsymbol{n})$$

is added to the integrals (3) and (4).

Thus, for this system, by Theorem 1 the flow is rectified after a change of time, and by Theorem 2 there is a Poisson structure reducing the system to a conformally Hamiltonian form.

In [15] (see also [1, 14]) a Poisson structure with annihilators (Casimir function) F_0, F_2 was found which in the variables

$$\boldsymbol{L} = \rho \boldsymbol{M}, \quad \boldsymbol{\gamma} = \boldsymbol{n}, \quad \rho = \big(D^{-1} - (\boldsymbol{n}, \mathbf{A}\boldsymbol{n})\big)^{-\frac{1}{2}}$$

looks as follows:

$$\{L_i, L_j\} = \varepsilon_{ijk}\bigg(L_k - \rho^2(\boldsymbol{L}, \boldsymbol{\gamma})\frac{\gamma_k}{J_k}\bigg), \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0;$$

furthermore, the equations of motion are represented in the conformally Hamiltonian form

$$\dot{L}_k = \rho\{H, L_k\}, \quad \dot{\gamma}_k = \rho\{H, \gamma_k\}, \quad k = 1, 2, 3,$$
$$H = \frac{1}{2}\Big(\big(D^{-1} - (\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})\big)(\boldsymbol{L}, \mathbf{A}\boldsymbol{L}) + (\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{L})^2\Big).$$

Example 2. We now consider the system (2) for $U = 0$ and $k = -1$; in this case there is an additional linear integral of the form

$$F_2 = (\widetilde{\mathbf{J}}\boldsymbol{M}, \boldsymbol{n}), \quad \widetilde{\mathbf{J}} = \mathrm{diag}(J_2 + J_3 - J_1, J_3 + J_1 - J_2, J_1 + J_2 - J_3),$$

so that here Theorem 1 can also be applied and the solution of the equations of motion theoretically can be obtained by quadratures. However, explicit quadratures by using separation of variables were found only for $F_2 = 0$ (see [16]). The question of explicit integration and Hamiltonization for $F_2 \neq 0$ still remains open.

We now describe how the system is reduced by quadratures [16] to a conformally Hamiltonian form. The corresponding separating variables z_1, z_2 are defined by the following relations:

$$\begin{gathered} n_i^2 = \frac{1}{G(z_1, z_2)} \frac{\det \mathbf{I}}{(J_i - d)J_jJ_k} \frac{(a_i - z_1)(a_i - z_2)}{(a_i - a_j)(a_i - a_k)}, \qquad (i, j, k) = (1, 2, 3), \\ M_i = \frac{n_j n_k}{2}(J_k - J_j)\left[\frac{\dot{z}_1}{(1 - a_j^{-1}z_1)(1 - a_k^{-1}z_1)z_2} + \frac{\dot{z}_2}{(1 - a_j^{-1}z_2)(1 - a_k^{-1}z_2)z_1}\right], \end{gathered} \tag{10}$$

where

$$G(z_1, z_2) = 1 - d(\mathrm{Tr}\,\mathbf{J} - 2d)(z_1 + z_2) + d(4\det\mathbf{J} - d\ \mathrm{Tr}(\mathbf{J}\widetilde{\mathbf{J}}))z_1z_2 \tag{11}$$

and $a_i = ((J_j + J_k - J_i)J_i)^{-1}$. The energy integral (3) and the square of the angular momentum (4) are represented in the form

$$\begin{aligned} H &= (z_1 - z_2)\frac{\det \mathbf{I}}{4G^2(z_1, z_2)}\left[\frac{\Psi(z_2)}{\Phi(z_1)z_2^2}\dot{z}_1^2 - \frac{\Psi(z_1)}{\Phi(z_2)z_1^2}\dot{z}_2^2\right], \\ F_1 &= (z_1 - z_2)\frac{\det \mathbf{I}}{4G^2(z_1, z_2)}\left[\frac{\psi(z_2)}{\Phi(z_1)z_2^2}\dot{z}_1^2 - \frac{\psi(z_1)}{\Phi(z_2)z_1^2}\dot{z}_2^2\right], \end{aligned} \tag{12}$$

where

$$\Psi(z) = d \det \mathbf{A} z^2 - \mathrm{Tr}(\mathbf{AJ})z + 2, \quad \psi(z) = (4\det \mathbf{J} - d\ \mathrm{Tr}(\mathbf{AJ}))z - (\mathrm{Tr}\,\mathbf{J} - 2d).$$

After the change of time

$$dt = \frac{1}{\sqrt{G(z_1, z_2)}}d\tau \equiv \sqrt{\frac{\det \mathbf{J}}{\det \mathbf{I}}(\mathbf{I}\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}\,d\tau, \tag{13}$$

on a level of the integrals $H = h$ and $F_1 = f$ the equations of motion are represented in the form

$$\begin{gathered} \frac{dz_1}{d\tau} = \frac{z_2\sqrt{R(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{d\tau} = \frac{z_1\sqrt{R(z_2)}}{z_2 - z_1}, \\ R(z) = -(z - a_1)(z - a_2)(z - a_3)\,(f\Psi(z) + h\psi(z)). \end{gathered} \tag{14}$$

On the manifold $\mathcal{M}^4 = \{F_0 = 1, F_2 = 0\}$, we choose the independent variables z_1, z_2, h, f and define a Poisson structure (of rank 4) as follows:

$$\begin{gathered} \{z_1, z_2\} = 0, \quad \{h, f\} = 0, \\ \{z_1, h\} = \frac{\Psi(z_2)}{\Delta}\sqrt{P(z_1)}, \quad \{z_2, h\} = -\frac{\Psi(z_1)}{\Delta}\sqrt{P(z_2)}, \\ \{z_1, f\} = -\frac{\psi(z_2)}{\Delta}\sqrt{P(z_1)}, \quad \{z_2, f\} = \frac{\psi(z_1)}{\Delta}\sqrt{P(z_2)}, \\ \Delta = \Psi(z_1)\psi(z_2) - \Psi(z_2)\psi(z_1), \quad P(z) = \frac{\Psi^2(z)}{z^2}R(z). \end{gathered}$$

In this case, the equations of motion (14) are represented in the conformally Hamiltonian form

$$\begin{gathered} \frac{dz_k}{d\tau} = \lambda(z_1, z_2)\{z_k, H\}, \quad H = h, \\ \lambda(z_1, z_k) = \frac{\Psi(z_1)\Psi(z_2)(z_1 - z_2)}{\Delta z_1 z_2}. \end{gathered}$$

A conformally Hamiltonian representation in the original physical variables $(\boldsymbol{M}, \boldsymbol{n})$ was found in [17].

2. HAMILTONIZATION IN A NEIGHBORHOOD OF A PERIODIC TRAJECTORY

In this section we consider systems that are not integrable; chaotic behavior is typical for them. In [18–21] there were given several examples of (non-holonomic) non-integrable systems, which can be reduced to a flow on a four-dimensional manifold that preserves the measure and has a first integral. For qualitative analysis of such systems, a Poincaré section is often used, by using which a two-dimensional map preserving area (in suitable variables) is constructed on a level surface of the first integral. Thus the analysis of this flow reduces to studying a family of two-dimensional maps parameterized by the value of the constant of the first integral. Such a representation makes it possible to use well-known (numerical) methods of chaos analysis, which are used as a rule for Hamiltonian systems with $1\frac{1}{2}$ and 2 degrees of freedom.

On the other hand, it is known [22] that every two-dimensional area-preserving map (homotopic to the identity map) can be analytically embedded into a Hamiltonian flow (possibly, with multi-valued Hamiltonian).

The natural question arises: *can the original system on the four-dimensional manifold be represented in a Hamiltonian form where the first integral is taken as the Hamiltonian*?

This question is obviously interesting not locally but either globally on the entire manifold, or in a neighborhood of some invariant submanifold. At the same time, it was pointed out above (see also [12]) that in integrable non-holonomic systems there are obstructions to representation of the system in a Hamiltonian form without change of time. Therefore in this section we consider the problem of representation of the original system (on a four-dimensional manifold) in a conformally Hamiltonian form in a neighborhood of a periodic solution.

Theorem 3. *Suppose that we have a dynamical system on a four-dimensional manifold that has an invariant measure μ and a first integral H. Let γ be a closed trajectory of this system on which $dH \neq 0$. Then in a neighborhood of this trajectory the system is conformally Hamiltonian with a Poisson structure of rank* 4.

Proof. The proof also uses a number of technical assertions. We fix a point on the trajectory γ and consider a four-dimensional neighborhood of it. The first step of the proof is introduction of convenient coordinates in this neighborhood.

It is clear that, from the topological viewpoint, the neighborhood of the trajectory γ is the direct product of a three-dimensional disc by a circle. Obviously, the flow on this neighborhood can be modeled as follows. First consider the cylinder $D^3 \times \mathbb{R}$ with coordinates (p, q, h) on the disc D^3 (where h is the value of the first integral H) and with coordinate t on the line $\mathbb{R}$. Then we identify on it the points with coordinates

$$(p, q, h, t) \quad \text{and} \quad (\Phi(p, q, h), t + f(p, q, h)), \tag{15}$$

where $\Phi\colon D^3 \to D^3$ is the three-dimensional Poincaré map and $f(p, q, h)$ is the return time of the trajectory that started from a point (p, q, h) located on the three-dimensional Poincaré section $\{t = 0\}$ (see Fig. 5). In the local coordinates under consideration, the vector field is written in the form $\boldsymbol{v} = \frac{\partial}{\partial t}$.

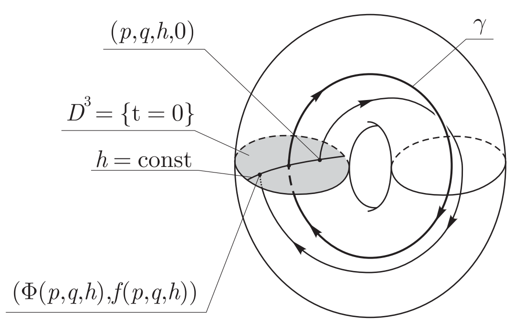


Fig. 5. Schematic depiction of a *four-dimensional* neighborhood of the periodic trajectory γ and illustration of the *three-dimensional* Poincaré map described in the proof (the boundary of the neighborhood depicted in the form of a torus is not invariant under the flow in the general case).

Thus, we have introduced the coordinates (p, q, h, t) in the neighborhood. Of course, they satisfy some periodicity condition. It is the identification rule (15) that expresses this condition.

The fact that the system has the first integral H implies that the three-dimensional Poincaré map is written in the form

$$\begin{aligned} \widetilde{p} &= \widetilde{p}(p, q, h), \\ \widetilde{q} &= \widetilde{q}(p, q, h), \\ \widetilde{h} &= h, \end{aligned}$$

where $(\widetilde{p}, \widetilde{q}, \widetilde{h}) = \Phi(p, q, h)$ are the coordinates of the image of a point under the Poincaré map.

The preservation of the measure implies that for every fixed h the two-dimensional Poincaré map preserves area on every two-dimensional section $\{h = \text{const}, t = 0\}$. We can assume without loss of generality that p, q are canonical coordinates and

$$\widetilde{p} = \widetilde{p}(p, q, h), \quad \widetilde{q} = \widetilde{q}(p, q, h)$$

is a family of symplectic diffeomorphisms, where h plays the role of a parameter. One can thus say that in a neighborhood of the trajectory under consideration our dynamical system is completely characterized by this family of symplectic maps Φ_h and in addition by the (positive) function $f(p, q, h)$, which, however, can vary absolutely arbitrarily under a change of the parameter on the trajectories (therefore in the end it should not play any role in our constructions, since we are interested in the Hamiltonianity up to a conformal factor). We point out that under a variation of the parameter t on the trajectories the coordinates (p, q, h) remain the same.

Thus, we have a good model describing the behavior of the system in a neighborhood of the periodic trajectory under consideration with the local coordinates already introduced.

The proof of the theorem consists in an explicit construction of a Poisson structure (more precisely, a symplectic form) and uses yet another technical assertion.

Lemma 2. *Suppose that we have a family of symplectic maps* $\Phi_h\colon D^2 \to D^2$:

$$\widetilde{p} = \widetilde{p}(p, q, h), \quad \widetilde{q} = \widetilde{q}(p, q, h)$$

where (p, q) *are the canonical coordinates. Then this map can be extended to a four-dimensional symplectic map of the form*

$$\begin{aligned} \widetilde{p} = \widetilde{p}(p, q, h), \quad \widetilde{q} = \widetilde{q}(p, q, h), \\ \widetilde{h} = h, \quad \widetilde{t} = t + \phi(p, q, h). \end{aligned} \tag{16}$$

In other words, there exists a smooth (*analytic*) *function* ϕ *such that* (16) *is a symplectic map in the sense of the symplectic structure* $\boldsymbol{\omega} = dp \wedge dq + dh \wedge dt$.

A more general assertion can be stated.

Lemma 3. *Suppose that we have a family of symplectic maps* $\Phi_h\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *smoothly depending on a parameter* h.

Then this map can be extended to a $(2n+2)$*-dimensional symplectic map of the space* $\mathbb{R}^{2n+2} = \mathbb{R}^{2n}(x) \times \mathbb{R}^2(h, t)$ *in the following sense: there exists a smooth function* $\phi(\boldsymbol{x}, h)$, $\boldsymbol{x} \in \mathbb{R}^{2n}$, $h \in \mathbb{R}$, *such that the map*

$$\begin{aligned} \widetilde{x} = \Phi_h(\boldsymbol{x}), \\ \widetilde{h} = h, \quad \widetilde{t} = t + \phi(\boldsymbol{x}, h) \end{aligned} \tag{17}$$

is symplectic in the sense of the natural symplectic structure $\Omega = \boldsymbol{\omega} + dh \wedge dt$ *on the space* $\mathbb{R}^{2n+2} = \mathbb{R}^{2n}(\boldsymbol{x}) \times \mathbb{R}^2(h, t)$.

Proof. The proof consists in writing out a differential equation for $\phi(\boldsymbol{x}, h)$ and verifying that it always has solutions. Here, it is convenient to pass from the symplectic structure to the Poisson bracket. Since the transformation under consideration is symplectic with respect to the variables $\boldsymbol{x}$ by construction, it is easy to see that a necessary and sufficient condition for the symplecticity of the transformation (17) takes the form

$$\{\widetilde{x}_k, \widetilde{t}\} = 0,$$

where $\widetilde{x}_k = \widetilde{x}_k(\boldsymbol{x}, h)$ denotes one of the coordinate functions defining the family of transformations Φ_h.

If we denote by $\{\ ,\ \}_0$ the original Poisson bracket defined on the space $\mathbb{R}^{2n}(\boldsymbol{x})$, then these relations can be rewritten in the form

$$\{\widetilde{x}_k, \phi\}_0 + \frac{\partial \widetilde{x}_k}{\partial h} = 0.$$

It is easy to see that the second term in this equation is the derivative of the function $\widetilde{x}_k$ along a certain vector field $\boldsymbol{\xi}$, which is determined from the equation

$$\frac{\partial \widetilde{x}_k}{\partial h} = \boldsymbol{\xi}(\widetilde{x}_k) = \sum_{\alpha} \xi^{\alpha} \frac{\partial \widetilde{x}_k}{\partial x_{\alpha}}.$$

Hence,

$$\xi^{\alpha} = \sum_{k} \frac{\partial x_{\alpha}}{\partial \widetilde{x}_k} \frac{\partial \widetilde{x}_k}{\partial h}$$

or, which is the same,

$$\boldsymbol{\xi} = \frac{d}{d\varepsilon}\Big|_{\epsilon=0} \Phi_h^{-1}(\Phi_{h+\epsilon}(\boldsymbol{x})).$$

Thus, the differential equation for the function ϕ has the form

$$\{\widetilde{x}_k, \phi\}_0 + \boldsymbol{\xi}(\widetilde{x}_k) = 0. \tag{18}$$

But this condition precisely means that the vector field $\boldsymbol{\xi}$ is Hamiltonian and the function ϕ is its Hamiltonian. Thus, the solubility of equation (18) is equivalent to the Hamiltonianity of the vector field $\boldsymbol{\xi}$. But its Hamiltonianity follows from the following well-known fact [23, 24]. If Ψ_{ε} is a family of symplectic maps (in this case, $\Psi_{\varepsilon} = \Phi_h^{-1} \circ \Phi_{h+\varepsilon}$) such that $\Psi_0 = \mathrm{id}$, then its derivative with respect to the parameter at zero is a Hamiltonian vector field (here we do not distinguish between local and global Hamiltonianity, since all the neighborhoods under consideration are simply connected). The lemma is proved.

Now, to complete our proof we need to suitably change the parameter on the trajectories, that is, time t. Clearly, from the viewpoint of the map (15), a change of the parameter t causes only a change in the function $f(p, q, h)$. Moreover, it is easy to see that this function can be made absolutely arbitrary. In particular, by a re-parametrization we can make sure that it becomes equal to $\phi(p, q, h)$, that is, the map (15) becomes symplectic with respect to the symplectic structure written in the canonical form $\boldsymbol{\omega} = dp \wedge dq + dh \wedge dt$. We point out that the required re-parametrization can be easily made analytic (see more details in Appendix).

It now remains only to consider the obtained result from a somewhat different viewpoint. In the neighborhood of the closed trajectory under consideration we have some system of coordinates (p, q, h, t) determined modulo the transformation (15), which is none other than the four-dimensional Poincaré map for our periodic trajectory. Since this map is symplectic, $\boldsymbol{\omega} = dp \wedge dq + dh \wedge dt$ defines a well-defined symplectic structure in our neighborhood.

It remains only to observe that the vector field $\boldsymbol{v}$ (we keep the previous notation for the re-parameterized field) has the form $\dfrac{\partial}{\partial t}$, which coincides with the Hamiltonian vector field of the function $H = h$, as required. The theorem is proved. □

Example 3 (Hamiltonianity of the rubber ball on a sphere). In this case, we consider as an example the rubber ball on a sphere described by the system (5) for $U \equiv 0$ (we point out that for the ordinary ball (2) there arise obstructions to Hamiltonization in the neighborhood of periodic orbits, which are discussed below, see Section 4). Then the system (5) admits simplest periodic solutions of the form

$$\begin{gathered}\sigma_k\colon n_i = \cos\varphi(t),\ n_j = \sin\varphi(t),\ n_k = 0,\ \omega_i = \omega_j = 0,\ \omega_k = \omega_0,\\ \varphi(t) = k\omega_0 t + \varphi_0, \quad \varphi_0, \omega_0 \text{ are const.}\end{gathered} \tag{19}$$

The differentials of the integrals (7) and (8), for example, for the solution σ_3 are determined as

$$\begin{gathered}dF_0 = 2\cos\varphi\, dn_1 + 2\sin\varphi\, dn_2, \quad dF_1 = \omega_0\, dn_3 + \cos\varphi\, d\omega_1 + \sin\varphi\, d\omega_2,\\ dH = J_3\omega_0\, d\omega_3,\end{gathered}$$

and are obviously linearly independent. Consequently, the solutions (19) are nondegenerate (that is, dH is nonzero modulo the differentials dF_0, dF_1). And indeed, as shown in [14], there exists a Poisson structure with the annihilators F_0, F_1 that reduces the system (5) to a conformally Hamiltonian form.

The corresponding Poisson structure has the simplest form in the following variables:

$$\boldsymbol{M} = \frac{\sqrt{\det \mathbf{J}}}{k}(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{\frac{1}{2k}}\mathbf{J}^{\frac{1}{2}}\boldsymbol{\omega}, \quad \boldsymbol{\gamma} = (\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{-\frac{1}{2}}\mathbf{J}^{-\frac{1}{2}}\boldsymbol{n};$$

then

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{20}$$

and the Hamiltonian is the energy integral (8)

$$H = \frac{1}{2}(\mathbf{J}\boldsymbol{\omega}, \boldsymbol{\omega}) = \frac{1}{2}\frac{k^2}{\det \mathbf{J}}(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{1/k}\boldsymbol{M}^2; \tag{21}$$

furthermore,

$$\dot{M}_i = (\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma})^{1-\frac{1}{2k}}\{M_i, H\}, \quad \dot{\gamma}_i = (\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma})^{1-\frac{1}{2k}}\{\gamma_i, H\}.$$

The bracket (20) is the Lie–Poisson bracket of the (co)algebra $e(3)$. The functions $(\boldsymbol{M}, \boldsymbol{\gamma})$, $(\boldsymbol{\gamma}, \boldsymbol{\gamma})$ are its Casimir functions.

Remark. As shown in [14], for $k = 1$ this system is isomorphic to the integrable system of Veselova [25], for which a conformally Hamiltonian representation was for the first time found in [26].

3. HAMILTONIZATION IN A NEIGHBORHOOD OF AN EQUILIBRIUM

The question of a criterion for conformal Hamiltonianity in a neighborhood of an equilibrium is, apparently, fairly difficult and still has not been solved. Here we present a number of observations relating to this problem.

It is easy to see that in this case there is the following obvious necessary condition.

Proposition 3. *Let $\boldsymbol{x}_0$ be a fixed point of a vector field $\boldsymbol{v}(\boldsymbol{x})$. If $\boldsymbol{v}$ is conformally Hamiltonian in a neighborhood of this point and the corresponding Poisson structure $\mathbf{J}$ has rank $2k$ at this point, then the linearization of this vector field has at most $2k$ nonzero eigenvalues, and these eigenvalues occur in pairs $\lambda, -\lambda$.*

Proof. The proof of this assertion is obvious.

It is easy to construct an example of a dynamical system on $\mathcal{M}^4$ with an invariant measure and two integrals for which the condition indicated above does not hold.

Example 4. Let $\boldsymbol{v}(\boldsymbol{x}) = (x_1, 2x_2, 2x_3, -4x_4)$. This vector field preserves the measure $\mu = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ and has two polynomial integrals $f_1 = x_1^4 x_4$ and $f_2 = x_2 x_3 x_4$. Proposition 3 shows that this vector field is not conformally Hamiltonian.

Comment. Obstructions to the existence of an invariant measure for non-holonomic systems near a fixed point were considered by Kozlov [27] (then certainly there does not exist a conformally Hamiltonian representation). He found a simplest condition on the eigenvalues λ_i of the linearization matrix of the form $\sum\limits_i \lambda_i = 0$ that is necessary for the existence of a measure with analytic density (this condition is weaker than in Proposition 3). However, apparently this condition is not sufficient: for example, in [20] a number of non-holonomic systems were found (for example, balanced ellipsoids whose dynamical and geometric axes coincide) for which the condition $\sum\limits_i \lambda_i = 0$ holds. Nevertheless, so far neither a measure, nor, all the more, a conformally Hamiltonian representation have been found for them. In [28] conditions were found under which a rolling inhomogeneous ellipsoid has no invariant measure.

Comment. For the Routh problem (discussed in [18]) that is related to stability analysis of a permanent rotation of a homogeneous ball at an apex of an elliptic paraboloid in the gravity field (and for the non-holonomic Jacobi problem [21]), also no conformally Hamiltonian representation is known so far. However, by applying Moser's theorem on the existence of invariant curves of an area-preserving map of an annulus, one can indicate a sufficient condition for the stability of these rotations and thus extend KAM theory to the case of conservative systems having an invariant measure.

4. OBSTRUCTIONS TO HAMILTONIZATION

We now consider the question of possible obstructions to representation of a system in a conformally Hamiltonian form.

Apart from the above-mentioned obstructions, in non-holonomic systems degenerate periodic orbits can occur in a neighborhood of an equilibrium, which also obstruct representation of the system in a conformally Hamiltonian form. Below we also describe global obstructions, which have topological nature. However, this question practically has not yet been studied in full measure.

4.1. Degenerate Periodic Orbits

We assume that on an n-dimensional manifold $\mathcal{M}^n$ we are given a system

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}) \tag{22}$$

preserving an invariant measure which also has a set of first integrals $f_1, \ldots, f_k$ satisfying the condition that $n - k + 1 = 2m$ is an even number.

Remark. As a rule, in available examples (see [1, 14]), on a six-dimensional manifold $\mathcal{M}^6$ a system is given with three first integrals f_1, f_2, f_3 and an invariant measure.

The simplest obstruction when it is certainly impossible to form Casimir functions and a Hamiltonian for the system (22) from the integrals $f_1, \ldots, f_k$ is described as follows.

Proposition 4. *If there exists a solution of the system $\gamma(t)$ that is not a fixed point and satisfies the condition*

$$\operatorname{rank}(df_1,\, \ldots,\, df_k)|_{\gamma(t)} < k, \tag{23}$$

then in a neighborhood of $\gamma(t)$ there does not exist a skew-symmetric representation of the system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}) = \mathbf{J}(\boldsymbol{x})\, dH(x), \quad H(x) = H(f_1(x),\, \ldots,\, f_k(x)), \tag{24}$$

with a skew-symmetric matrix $\mathbf{J}(\boldsymbol{x}) = -\mathbf{J}^{\mathrm{T}}(\boldsymbol{x})$ for which $\operatorname{rank}\mathbf{J}(\boldsymbol{x}) = n - k + 1$ and $\operatorname{Ker}\mathbf{J}(\boldsymbol{x})$ belongs to the linear hull of $df_1,\, \ldots,\, df_k$.

Proof. The proof reduces to verifying that under condition (23) and for an arbitrary choice of $\mathbf{J}(\boldsymbol{x})$ and $H(x)$ satisfying the conditions stated, the relation

$$\mathbf{J}(\boldsymbol{x})\, dH(x)|_{\gamma(t)} \equiv 0$$

holds, that is, $\gamma(t)$ cannot be a solution distinct from a fixed point. □

Remark. In this assertion, the only case that is excluded is where the hyperplane $\operatorname{Ker}\mathbf{J}(\boldsymbol{x})$ must be everywhere tangent to a level surface of the Casimir functions $\mathcal{M}^{2m} = \{F_1(f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})) = c_1, \ldots, F_{k-1}(f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})) = c_{k-1}\}$. Here, the question of whether a conformally Hamiltonian representation is possible with tensor $\mathbf{J}(\boldsymbol{x})$ of higher or lower rank remains open.

Example 5. For $U = 0$, the equations of motion (2) admit three families of closed trajectories defined by the relations

$$\sigma_k\colon n_i = \sin\varphi, \quad n_j = \cos\varphi, \quad n_k = 0, \quad M_i = M_j = 0, \quad M_k = c, \quad i \neq j \neq k \neq i. \tag{25}$$

For the periodic trajectory σ_3 we find

$$dF_0 = 2\cos\varphi\, dn_1 + 2\sin\varphi\, dn_2, \quad dF_1 = 2c\, dM_3, \quad dH = a_3 c\, dM_3.$$

Thus, in this case, $\operatorname{rank}(dF_0, dF_1, dH)|_{\sigma_k} = 2$; consequently, by Proposition 4, *in a neighborhood of the periodic solutions σ_k, it is impossible to represent the system* (2) *in a conformally Hamiltonian form where the Casimir functions and the Hamiltonian are functions of the first integrals* (3)*.*

Remark. At the same time, the system (2) can be represented in a skew-symmetric form as follows:

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}}, \qquad \dot{\boldsymbol{n}} = \frac{k}{d\Lambda}\boldsymbol{n} \times \frac{\partial H}{\partial \boldsymbol{n}}, \tag{26}$$

where the integrals F_0, F_1 are annihilators of the corresponding matrix $\mathbf{J}(\boldsymbol{M}, \boldsymbol{n})$. Nevertheless, as we see, $\Lambda(\boldsymbol{M}, \boldsymbol{n})|_{\sigma_k} = 0$; consequently, this representation *has a singularity* on the periodic solutions (25).

Furthermore, for the skew-symmetric representation (26), no reducing factor is known (that is, a function $K(\boldsymbol{M}, \boldsymbol{n})$ such that $K\mathbf{J}$ satisfies the Jacobi identity). Consequently, the question of conformal Hamiltonianity of the system far from the periodic solutions (25) is also unsolved.

4.2. Global obstructions

If the question is posed globally, that is, an attempt is made to carry out Hamiltonization of a dynamical system as a whole on the manifold $\mathcal{M}^4$, then it is easy to see that there exist obstructions. We present a simple example showing of what type these can be.

Example 6. As $\mathcal{M}^4$ we consider the four-dimensional sphere $S^4 = \{x_1^2 + \dots + x_5^2 = 1\} \subset \mathbb{R}^5$.

We consider the vector field $\boldsymbol{\xi} = (\lambda x_2, -\lambda x_1, \mu x_4, -\mu x_3, 0)$.

It is easy to see that this vector field has an invariant measure induced on the sphere by the standard volume form $dx_1 \wedge \dots \wedge dx_5$, and two first integrals x_5 and $x_1^2 + x_2^2$.

Locally, this vector field can be easily Hamiltonized (even without change of time), but no suitable global symplectic structure exists on S^4 simply because the four-dimensional sphere S^4 is not a symplectic manifold [29].

One can give an example showing that obstructions to Hamiltonization exist even in a neighborhood of a two-dimensional 'integral manifold' $\mathcal{X} = \{H = h_0, f = f_0\}$, but in the case where $\mathcal{X}$ has singularities. The idea of such an example uses one not quite obvious property of an integrable Hamiltonian system.

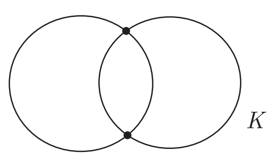
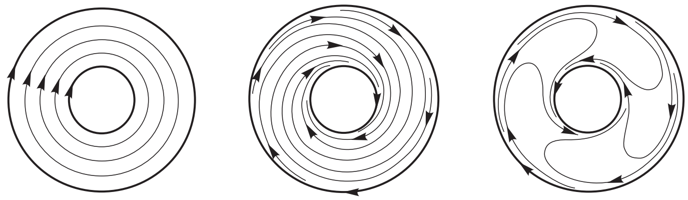


Fig. 6

A typical singular integral manifold is the Cartesian product of some graph K (see Fig. 6) by a circle (see, for example, [30]). The vertices of the graph correspond to the periodic trajectories of the system, and the (open) edges to the separatrix manifolds. The behavior of a Hamiltonian vector field on a separatrix manifold, which is an annulus from the topological viewpoint, can be of three different types. Either all the trajectories are closed, or they are asymptotic to the boundary circles of the annulus, which, in turn, can be oriented in the same or opposite direction (see Fig. 3).

We consider the last situation (Fig. 3c). It can be easily realized on a singular leaf of the type $\mathcal{X} \simeq K \times S^1$, where K is the graph with two vertices and four edges depicted in Fig. 6 (in the book [30] it is called C_2). Here, the singular trajectories corresponding to the vertices of the graph are oriented in the opposite fashion. It is known (see [30], Proposition 3.10) that if a singular integral surface of this type occurs in an integrable system, then this surface is unstable, that is, it decomposes into two simpler integral surfaces for an arbitrarily small change of the energy level of the system. However, if a priori there is not any symplectic structure, then nothing prevents us from constructing the vector field as follows.



(a) (b) (c)

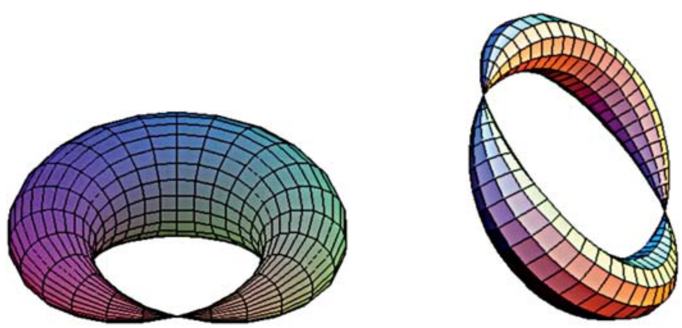
Fig. 7.

Example 7. First we consider an actual integrable Hamiltonian system with a singular integral surface $\mathcal{X} \simeq K \times S^1$ with the Hamiltonian vector field described above (that is, as in Fig. 3c). An example of construction of an integrable system with this dynamics on a singular leaf can be found in [30]. We fix the corresponding level of the Hamiltonian $\mathcal{Q}^3 = \{H = h_0\}$ and restrict our vector field $\boldsymbol{v}$ to $\mathcal{Q}^3$. We forget altogether about the behavior of this vector field outside $\mathcal{Q}^3$. We now consider the new manifold $\mathcal{M}^4 = \mathcal{Q}^3 \times (-\varepsilon, \varepsilon)$. Let $t \in (-\varepsilon, \varepsilon)$ be an additional coordinate. We extend naturally the additional integral f and the function H to $\mathcal{M}^4$ by setting $f(\boldsymbol{x}, t) = f(\boldsymbol{x})$ and $H(\boldsymbol{x}, t) = t$, where $x \in \mathcal{Q}^3$ and $t \in (-\varepsilon, \varepsilon)$. The vector field $\boldsymbol{v}$ can be trivially extended from $\mathcal{Q}^3$ to $\mathcal{M}^4 = \mathcal{Q}^3 \times (-\varepsilon, \varepsilon)$ in such a way that for every t it becomes tangent to $\mathcal{Q}^3_t = \mathcal{Q}^3 \times \{t\}$ and simply coincides with the original vector field under the natural identification of $\mathcal{Q}^3_t$ with $\mathcal{Q}^3$. In other words, the behavior of $\boldsymbol{v}$ on all levels $\{H = h_0\}$ is absolutely identical (for the original Hamiltonian vector field this was not the case, since the singular surface $\mathcal{X}$ had to decompose into two simpler ones!).

The vector field thus constructed has the two integrals f and H and, furthermore, preserves the natural measure $\mu \wedge dt$ on $\mathcal{M}^4$, where μ is the measure on $\mathcal{Q}^3$ that was preserved by the original vector field.

This vector field cannot be Hamiltonized, since for every $t \in (-\varepsilon, \varepsilon)$ the singular leaf by construction contains a separatrix manifold with dynamics shown in Fig. 3c and does not decompose as H changes, which is impossible in the case of Hamiltonian systems (see [30], Proposition 3.10).

Example 8 (Monodromy as an obstruction to Hamiltonization). The topological structure of the fibration of the phase space into integral manifolds in the neighborhood of a singular fiber can also be viewed as an obstruction to Hamiltonization. One of such obstructions has been found by R. Cushman and J. Duistermaat [31]. We briefly describe this construction.



(a) (b)

Fig. 8

Consider a dynamical system on $\mathcal{M}^4$ with an invariant measure and two first integrals H and F. In many integrable (both hamiltonian and non-holonomic) systems there appear equilibrium points of focus type. The singular integral surface $\mathcal{X}_{\text{sing}} = \mathcal{X}_{s_0}$ passing through such a point is a torus with one or several pinched points (Figs. 8a and 8b). This singular surface is isolated in the sense that all neighboring integral surfaces $\mathcal{X}_s$ are regular and diffeomorphic to a two-dimensional torus (while approaching the singular surface $\mathcal{X}_{f)\backslash\}}$ some "distinguished" cycles on these tori shrink and finally become the singular points).

In terms of the integral map $\Phi : \mathcal{M}^4 \to \mathbb{R}^2(H,F)$ and its bifurcation diagram $\Sigma \subset \mathbb{R}^2(H,F)$, this means that the image $\Phi(\mathcal{X}_{\mathrm{sing}})$, as a point $s_0 \in \Sigma$, is isolated so that all the other points $s \in U_\varepsilon(s_0) \setminus \{s_0\}$ are regular.

On each torus $\mathcal{X}_c$, we can consider a rotation number $\rho(s)$ which is well defined and is invariant under the time scaling (as soon as we fixed a pair of basis cycles on $\mathcal{X}_c$). It is clear that locally $\rho(s)$ is a smooth function of s (or, equivalently, of H and F) which may have a singularity at s_0 and usually does. Moreover, in many examples of this kind s_0 is a branch point of logarithmic type.

From the topological view point this means that the fibration into 2-dimensional tori over the punctured neighborhood of the singular point $s_0 \in \Sigma$ is not trivial (although it is locally trivial): a torus $\mathcal{X}_s$ moving around s_0 comes back to the initial position with some non-trivial transformation of basis cycles. This topological phenomenon, called *the monodromy*, is, in particular, an obstruction to the existence of single-valued action-angle variables over $U_\varepsilon(s_0) \setminus \{s_0\}$, see [32].

It is well known (see [33, 34]) that for focus types singularities of integrable Hamiltonian systems the behavior of ρ depends on the number k of singular points on $\mathcal{X}_{s_0}$. Namely, if the basis cycles on $\mathcal{X}_s$ are properly chosen then each turn around s_0 leads to the increment of ρ by $k \in \mathbb{Z}$. Roughly speaking, each focus point gives a contribution of 1 to the increment of ρ.

In [31] it was shown that the situation in the non-Hamiltonian case is somewhat similar but the principal difference is that each separate focus point gives a contribution of ± 1, where the sign $\pm$ may vary from one point to another (this sign is related to the choice of orientation which in the Hamiltonian case is uniquely determined by the symplectic structure and, therefore, this sign can consistently be chosen the same for all the focus points).

For example, for a doubly pinched torus (Fig. 8b) it may happen that the monodromy is trivial, i.e., $\Delta\rho = 1 + (-1) = 0$. Such a situation would be an obstruction to Hamiltonization in a neighborhood of $\mathcal{X}_{\mathrm{sing}}$, as in the Hamiltonian case we necessarily have $\Delta\rho = 1 + 1 = 2$.

In [31] the authors suggested an example of a non-holonomic system where such a scenario is likely to be realized. This is the dynamical system describing the rolling of a homogeneous ellipsoid on a plane (the non-holonomic constrain is that the velocity of the point of contact is zero). Unfortunately, in this paper there are no convincing enough arguments for this conjecture. In this system there is indeed a singular fiber $\mathcal{X}_{\mathrm{sing}}$ with two focus points (i.e., doubly pinched torus shown in Fig. 8b) but our preliminary considerations have led us to the opposite conclusion: the monodromy for this system will be the same as in the Hamiltonian case:

$$\rho \mapsto \rho + \Delta\rho = \rho + 2.$$

We think that this question is worth being studied and we hope to do it in the nearest future. In any case, finding a concrete example of a non-holonomic mechnical system in which the above described obstruction is realized remains an open and very interesting problem.

5. OPEN QUESTIONS

In conclusion we indicate some open questions, which may be solved by using the methods developed in this paper.

- Suppose that we have a dynamical system on a four-dimensional manifold $\mathcal{M}^4$ having two independent integrals H and f and an invariant measure μ. Then on a level surface $\mathcal{Q}^3 = \{H = \mathrm{const}\}$ there arises a foliation into tori with singularities, which in the case of general position must, as in the Hamiltonian case, be described by the Fomenko–Zieschang invariant, that is, by the so-called marked molecule [30, 35].

  The natural question arises about conformal Hamiltonianity of such a vector field in a neighborhood of an "isoenergetic" manifold $\mathcal{Q}^3$, that is, on a "molecule". Example 7 described above illustrates possible obstructions to Hamiltonization. Is it true that the absence of separatrix manifolds shown in Fig. 3c guarantees the conformal Hamiltonianity of the system?

This question seems quite soluble, since there is a well-developed theory of orbital classification on isoenergetic surfaces [36, 37], which describes in detail the behavior of the trajectories of integrable Hamiltonian systems on $\mathcal{Q}^3$.

- In applications, situations arise where a vector field on $\mathcal{M}^4$ has two independent integrals, but nothing is known about existence of an invariant measure [14]. From the purely theoretical viewpoint, it is fairly easy to construct examples of this type without an invariant measure (and, consequently, not conformally Hamiltonian). It suffices to make sure that a limit cycle should appear on an invariant torus with rational rotation number. It would be interesting to find out whether examples of this kind occur in actual mechanical systems.

APPENDIX

In the proofs of the main theorems of this paper, we used mentionings of certain "standard facts of differential geometry and topology". Here we present their rigorous statements.

For example, in the proof of Theorem 1 it is said that the smooth angular coordinate τ can be approximated by an analytic angular coordinate $\tilde{\tau}$. This is a special case of the following absolutely general assertion on approximation of smooth functions by analytic ones on compact manifolds.

Theorem. *Let $f\colon M \to \mathbb{R}$ be a C^∞-smooth function on a real-analytic compact manifold M. Then there exists a real-analytic function $g\colon M \to \mathbb{R}$ that approximates f together with all its derivatives up to and including order k to within any pre-assigned $\varepsilon > 0$.*

For the general construction of the proof of this theorem and its generalizations see [38].

Remark. If M is a circle, then the approximation is given by the Fourier series. In the case of an arbitrary manifold, an analogue of the Fourier series is, for example, the expansion in the eigenfunctions of the Laplace–Beltrami operator (see more details in [39]).

The second standard assertion is one of the steps in the proof of Liouville's theorem (see, for example, [40]), and therefore we present it without proof.

Lemma. *Suppose that on a torus we are given two commuting vector fields that are linearly independent at every point; then on the torus there exist angular coordinates in which these vector fields are rectified (that is, are represented in the form $\dot{\varphi}_1 = \omega_1$, $\dot{\varphi}_2 = \omega_2$, $\omega_i = \mathrm{const}$).*

Similarly, if we are given a family of tori $T^2 \times D^{n-2}$ and two commuting vector fields that are tangent to the tori and are linearly independent at every point, then angular coordinates in which the fields are rectified can be chosen in compatible fashion simultaneously on all at once tori. In the real-analytic situation, these angular coordinates are also real-analytic.

In conclusion we present the proof of the fact that the re-parametrization of time in Theorem 3 can be made analytic. This proof is based on the following assertion.

Lemma. *Consider a real-analytic dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ with a closed trajectory and its Poincaré map defined on some analytic transversal domain. It is possible to analytically re-parameterize the system in such a way that the return time to the area element becomes equal to 1 simultaneously for all at once trajectories.*

We now obtain the following: by this lemma any function $f(p, q, h)$ (which determines the return time) can be made equal to 1 by an analytic re-parametrization. Then a reverse re-parametrization will produce from 1 any pre-assigned function. Thus, by applying this lemma two times (first we obtain 1 from $f(p, q, h)$, and then obtain $\phi(p, q, h)$) from 1, we obtain the required analytic re-parametrization.

Proof of the lemma. Since we consider a periodic trajectory, in a neighborhood of it we can introduce an analytic angular coordinate τ (the same idea as in Theorem 1), which changes by 1 when going around the trajectory. The sections $\tau = \mathrm{const}$ are transversal to the flow. Therefore the coordinates p, q, h, τ are good local coordinates along the trajectory.

In this system of coordinates, the vector field ∂_τ and the original vector field $\boldsymbol{v} = \partial_t$ are collinear and analytic; this precisely means that one can be obtained from the other by an analytic reparametrization (or, which is the same, by an analytic conformal factor).

Thus, this lemma immediately follows from the fact that an analytic angular coordinate can be defined in a neighborhood of any closed analytic curve. The lemma is proved.

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