# Singularities of bi-Hamiltonian systems and stability analysis

(preliminary version of Lecture Notes)

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**Abstract**

Main goal of this minicourse is to explain how a bi-Hamiltonian structure can be used for the qualitative analysis of the dynamics of integrable systems.

## Content



## Acknowledgements

These lecture notes are based on the results of a joint project we have been working on together with Anton Izosimov and Andrey Oshemkov for several years.

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# Lecture 1. Integrable systems: Singularities and bifurcations

**Main ingredients:**

- Integrable systems
- Lagrangian fibration and its singularities
- Why singularities are important?
- Non-degenerate singularities and their basic properties

## Integrable systems, Lagrangian fibrations and their singularities

We consider a symplectic manifold $(M, \omega)$, i.e. a smooth manifold $M$ endowed with a closed non-degenerate differential 2-form $\omega$:

$$d\omega = 0 \quad \text{and} \quad \det(\omega_{ij}) \neq 0.$$

A Hamiltonian system on $M$ is understood as a system of ODEs (dynamical system) of the following form

$$\dot{x} = X_H(x) = \omega^{-1}(dH(x)) \tag{1}$$

or, in coordinates,

$$\dot{x}^i = \omega^{ij}(x)\frac{\partial H}{\partial x^j},$$

where $H : M \to \mathbb{R}$ is a smooth function called the *Hamiltonian* of the system, and $\omega^{ij}$ are the components of the inverse matrix to $\omega = (\omega_{jk})$, i.e. $\omega^{ij}\omega_{jk} = \delta^i_k$.

In what follows, we shall discuss a more general situation, namely, Hamiltonian systems on Poisson manifolds but for the purposes of this lecture it is more convenient to consider the symplectic case. The passage from "symplectic" to "Poisson" is more or less straightforward.

Among all Hamiltonian systems we distinguish a very special and important subclass, the so-called called *Liouville integrable* systems. This is the main subject of the course.

Recall that a Hamiltonian system (1) is said to be Liouville integrable if there exist $n$ smooth functions $f_1, \dots, f_n : M \to \mathbb{R}$ satisfying three properties:

- $f_1, \dots, f_n$ are first integrals of $X_H(x)$;
- $f_1, \dots, f_n$ pairwise commute w.r.t. the Poisson bracket on $M$;

- $f_1, \ldots, f_n$ are independent almost everywhere.

These conditions immediately imply that $M$ is foliated into common levels of integrals

$$L_a = \{f_1(x) = a_1, \ldots, f_n(x) = a_n\}, \quad a = (a_1, \ldots, a_n) \in \mathbb{R}^n$$

each of which is invariant under the flow of $X_H$.

The dynamics of the integrable Hamiltonian system on a regular fiber $L_a$ is described by the classical Liouville-Arnold theorem.

**Theorem 1** (Dynamics)**.** *Let $L_a$ be regular, compact and connected, then $L_a$ is an $n$-dimensional torus and the dynamics of $X_H$ on $L_a$ is quasi-periodic.*

In a sufficiently small neighbourhood of $L_a$, the foliation into common levels of the integrals is trivial, in particular, all neighboring fibers are tori with quasiperiodic dynamics. Moreover, the structure of this fibration is standard in the sense that locally in a neighbourhood of $L_a$ the fibration is symplectomorphic to the following canonical model. Let $M_{\text{reg}} = T^n \times D^n$ be the direct product of a torus $T^n$ and a disc $D^n$ endowed with the symplectic structure $\omega = \sum_{i=1}^n ds_i \wedge d\phi_i$ where $\phi_1, \ldots, \phi_n$ are angle coordinates on $T^n$ and $s_1, \ldots, s_n$ are coordinates on $D^n$. Then the functions $s_1, \ldots, s_n$ commute and define the (trivial) fibration of $M_{\text{reg}}$ into tori.

**Theorem 2** (Fibration)**.** *Let $L_a$ be regular, compact and connected. Then there exists a neighbourhood $U(L_a)$ and a fiberwise symplectomorphism from $U(L_a)$ to this canonical model $F : U(L_a) \to M_{\text{reg}}$.*

If $L_a$ consists of several regular components, then all of them are tori and the conclusion of the Liouville-Arnold theorem (Theorems 1 and 2) holds true for each of them separately.

By regularity we mean that the differentials $df_1(x), df_2(x), \ldots, df_n(x)$ are linearly independent at all points $x \in L_a$. However, for some fibers $L_b$ the regularity condition may fail, thus in fact we deal with a singular Lagrangian fibration whose fibers are, by definition, connected components of common levels of the commuting integrals $f_1, \ldots, f_n$.

**Remark 1.** This fibration is *Lagrangian* in the sense that all regular fibers $L_a$ are Lagrangian submanifolds, i.e., the restriction of $\omega$ on $L_a$ vanishes. Notice that every Lagrangian fibration is locally defined by a collection of commuting functions.

The main goal of the topology of integrable systems is to describe/classify such fibrations, their singularities and invariants.

From the viewpoint of Singularity Theory, the structure on the phase space $M$ we are interested in, is defined by the smooth map

$$\Phi = (f_1, \ldots, f_n) : M \to \mathbb{R}^n,$$

the so-called *momentum map*. In what follows, we always assume that $\Phi$ is proper.

The fibers are connected components of preimages $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$. Basically, we want to understand the structure of singularities of this map, bifurcations of regular fibers, global invariants and so on. However, in our setting there is one very essential difference from the usual Singularity Theory. The point is that the functions $f_1, \dots, f_n$ satisfy a very strong additional condition, namely, they *Poisson commute*. This property dramatically affects the structure of the singularities and of the fibration as a whole.

To clarify the structure of the fibration we introduce the set of critical points

$$\mathsf{S} = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$$

and the bifurcation diagram $\Sigma$ as the image of $\mathsf{S}$ under the momentum map:

$$\Sigma = \Phi(\mathsf{S}) \subset \mathbb{R}^n.$$

**Example 1.** .

Consider the Neumann system on $S^2$ that describes the motion of a point $x \in S^2$ in a quadratic potential.

The phase space for this system is the cotangent bundle $M^4 = T^*S^2$ considered as a symplectic manifold w.r.t. the canonical symplectic form $\omega = dx_1 \wedge dp_1 + dx_2 \wedge dp_2$.

In sphero-conical coordinates $(x_1, x_2)$, the Hamiltonian and first integral of the Neumann system are:

$$H = \frac{-P(x_1)p_1^2 + P(x_2)p_2^2}{x_1 - x_2} + x_1 + x_2,$$

$$F = \frac{x_2 P(x_1)p_1^2 - x_1 P(x_2)p_2^2}{x_1 - x_2} - x_1 x_2,$$

where $P(x) = (x+a)(x+b)(x+c)$.

So following the general scheme, we can define the momentum map

$$\Phi : T^*S^2 \to \mathbb{R}^2, \quad \Phi(x,p) = (H(x,p), F(x,p)),$$

and study its bifurcation diagram

$$\Sigma = \{\Phi(x,p) \mid \operatorname{rank} d\Phi(x,p) < 2\} \subset \mathbb{R}^2(H,F),$$

i.e., the image of the set of critical points $\mathsf{S}$.

This critical set $\mathsf{S}$ can be described by analysing the rank of the Jacobi matrix:

$$(x,p) \in \mathsf{S} \quad \text{if and only if} \quad \operatorname{rank} \begin{pmatrix} \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} & \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} \\ \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} & \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{pmatrix} < 2 \quad \text{at } (x,p) \in T^*S^2$$

Since, the variables $x_1$ and $x_2$ are separated, all computations in this case are straightforward and we come to the bifurcation diagram shown in Fig. 1. This bifurcation diagram helps to understand the structure of the corresponding Lagrangian fibration into invariant Liouville tori and its singularities. If we take a point $a \in \mathbb{R}^2$ in the image of the momentum map, which does not belong to the bifurcation diagram $\Sigma$, then its preimage $\Phi^{-1}(a)$ is a disjoint union of two-dimensional tori (how many?). If this point $a \in \mathbb{R}^2$ moves and crosses one of the branches of the bifurcation diagram, these tori undergo a certain bifurcation (which one?). The preimages of the points $P_1$, $P_2$ and $P_3$ can be understood as singularities of codimension two, they are somehow more singular than those located on the branches of $\Sigma$ (how to describe the topology of these singular fibers $\Phi^{-1}(P_i)$?).

These are typical questions we need to answer when doing the topological analysis of dynamics of integrable systems.

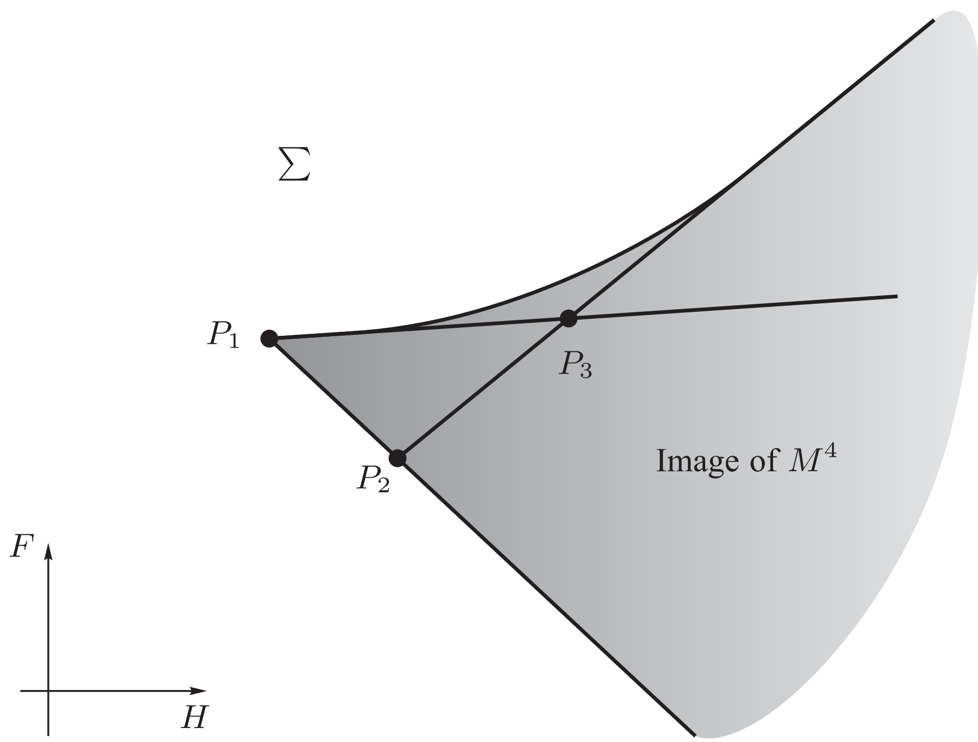


Figure 1: Bifurcation diagram for the Neumann system

In these lectures we will be interested in the singular set $\mathsf{S} \subset M$ and its properties (focusing mainly on local aspects). But why are these singularities important? Below is the list of areas in geometry, mechanics and mathematical physics where singularities of integrable systems play a very important role.

- Classical mechanics, topological analysis of integrable cases:

M.P.Kharlamov, A.A.Oshemkov, M.Odin, R.Cushman, L.Bates, A.V.Borisov, I.S.Mamaev

- Topological obstructions to integrability:

  V.V.Kozlov, A.T.Fomenko, I.A.Taimanov, G.Paternain, L.Butler

- Three-dimensional topology and the problem of isoenergy classification:

  A.T.Fomenko, H.Zieschang, S.V.Matveev, V.V.Sharko, A.V.Brailov

- Perturbation theory and topological invariants of dynamical systems:

  L.M.Lerman, Ya.L.Umanskii

- Theory of normal forms:

  L.Eliasson, J.Vey, H.Ito, Nguyen Tien Zung, E.Miranda

- Symplectic manifolds with actions of Lie groups, toric and almost toric manifolds:

  M.Atiyah, V.Guillemin, S.Sternberg, T.Delzant, M.Symington, M.Audin, San Vu Ngoc

- Quantization and global action-angle variables:

  J.Duistermaat, B.Zhilinskii, San Vu Ngoc, A.Pelayo

## Stability and singularities for integrable systems

One of the very important issues in the theory of dynamical systems is the stability of solutions. It turns out that for integrable systems "stability" and "singularities" are naturally related to each other. Two theorems below illustrate this relationship.

**Definition 1.** A fixed point $P_0$ of a dynamical system $v$ is called *stable*, if for any neighbourhood $U(P_0)$ there is a neighbourhood $V(P_0)$ such that every trajectory started in $V(P_0)$ does not leave $U(P_0)$.

Similarly for a periodic trajectory/solution: a periodic trajectory $\gamma(t)$ is called (orbitally) *stable*, if for any neighbourhood $U(\gamma)$ there is a neighbourhood $V(\gamma(0))$ such that every trajectory started in $V(\gamma(0))$ does not leave $U(\gamma)$.

Consider an integrable Hamiltonian system with two degrees of freedom on a symplectic manifold $(M^4, \omega)$ with a Hamiltonian $H$ and an additional integral $F$.

**Theorem 3.** *Let $\gamma(t)$ be a stable periodic solution of this system. Then $\gamma$ is singular, i.e., belongs to the singular set $\mathsf{S}$ (unless the system is resonant). Moreover, in the real analytic case $\gamma(t)$ is stable if and only if $\gamma(t)$ coincides with the common level of the integrals $H$ and $F$:*

$$\{\gamma(t), t \in \mathbb{R}\} = \{H(x) = H(x_0), F(x) = F(x_0)\}, \quad x_0 = \gamma(t_0).$$

For integrable systems of $n$ degrees of freedom with commuting integrals $f_1, \dots, f_n$ and momentum map $\Phi = (f_1, \dots, f_n) : M^{2n} \to \mathbb{R}^n$, we have

**Theorem 4.** *Let $P \in M^{2n}$ be an equilibrium point of a non-resonant integrable system. If $P$ is stable then $P$ is a critical point of $\Phi$, i.e., $P \in \mathsf{S}$ and, moreover,* $\operatorname{rank}\Phi(P) = 0$*, i.e., $P$ is a common equilibrium point for all the integrals $f_1, \dots, f_n$.*

These two results lead us to a strange conclusion that for stability analysis of integrable systems we do not need to consider the Hamiltonian equations. The only important thing is the momentum map and its singularities (or, equivalently, the corresponding singular Lagrangian fibration). In fact, this observation is very natural because the Liouville fibration contains almost all of the essential information about the dynamics of an integrable system.

## Non-degenerate singularities

What are the most typical singularities of integrable systems? The answer is given by the following definition. As above, we consider an integrable Hamiltonian system with commuting independent integrals $f_1, \dots, f_n$ on a symplectic manifold $(M^{2n}, \omega)$.

**Definition 2.** *Let $x \in M^{2n}$ be a singular point of rank zero, i.e., $df_i(x) = 0$, $i = 1, \dots, n$. It is called non-degenerate, if the operators $\omega^{-1}d^2 f_1, \dots, \omega^{-1}d^2 f_n$ generate a Cartan subalgebra in the symplectic Lie algebra $sp(2n, \mathbb{R}) = sp(T_x M, \omega)$.*

Equivalently, non-degeneracy means that the quadratic parts $d^2 f_1(x), \dots, d^2 f_n(x)$ are linearly independent and there is a linear combination $f = \sum a_i f_i$ such that the roots of the characteristic equation $p(\lambda) = \det(d^2 f(x) - \lambda \cdot \omega) = 0$ are all distinct.

The type of a non-degenerate singular point is defined by the type of the corresponding Cartan subalgebra. Different types of Cartan subalgebras (i.e. their conjugacy classes) are described by the Williamson theorem which can be interpreted as follows.

Consider three simple and quite typical singularities:

- elliptic type (centre): $f(p, q) = p^2 + q^2$,
- hyperbolic type (saddle): $f(p, q) = p^2 - q^2$,
- focus: $f_1 = p_1 q_1 + p_2 q_2$, $f_2 = p_1 q_2 - p_2 q_1$.

In general, we have a combination of these three types. In other words, in dimension $2n$ we can construct $n$ commuting functions by taking the functions of the above type which depend on different groups of variables (i.e., either pairs $(p_i, q_i)$ or quadruples $(p_i, q_i, p_{i+1}, q_{i+1})$). The linearisations of the corresponding Hamiltonian vector fields, i.e. operators $A_k = \omega^{-1} d^2 f_k$, define a Cartan subalgebra in $sp(2n, \mathbb{R})$.

Any other Cartan subalgebra of $sp(2n, \mathbb{R})$ is conjugate to one these "canonical" Cartan subalgebras.

Alternatively, one can say that the type of a Cartan subalgebra $\mathfrak{h} \subset sp(2n, \mathbb{R})$ is defined by the eigenvalues of a generic element $A \in \mathfrak{h}$. Its eigenvalues can be naturally divided into groups:

- a pair of imaginary numbers $i\beta$, $-i\beta$;
- a pair of real numbers $\alpha$, $-\alpha$;
- four complex numbers $\alpha + i\beta$, $\alpha - i\beta$, $-\alpha + i\beta$, $-\alpha - i\beta$.

The Williamson type of $\mathfrak{h}$ is just the triple $(k_e, k_h, k_f)$ with $k_e, k_h$ and $k_f$ being the number of imaginary, real and complex groups respectively.

**Theorem 5** (Vey, Eliasson). *The type of a singularity is its complete topological, smooth and even symplectic invariant. In other words, two singularities having the same Williamson type are locally symplectomorphic.*

From the topological point of view, this theorem states that every non-degenerate singularity can be represented as the product of the simplest singularities, i.e., centre, saddle and focus.

This statement remains true for singularities of an arbitrary rank. Moreover, it holds true in a neighbourhood of a non-degenerate orbit of the Poisson action generated by the commuting integrals $f_1, \dots, f_n$ (Nguyen Tien Zung, E.Miranda).

It is an important and fundamental fact that non-degenerate singularities admit a very natural not only local, but also semi-local description. To explain this construction, we first consider "elementary blocks"

- elliptic singularity $M^{\rm ell}$,
- hyperbolic singularity $M^{\rm hyp}$,
- focus singularity $M^{\rm foc}$,

each of which represents a (semilocal) singularity in dimension 2 or 4. Examples are shown in Fig. 2, 3 and 4. Starting from these singularities and taking the direct product of them

$$\underbrace{M_1^{\rm ell} \times \cdots \times M_{k_e}^{\rm ell}}_{k_e} \times \underbrace{M_1^{\rm hyp} \times \cdots \times M_{k_h}^{\rm hyp}}_{k_h} \times \underbrace{M_1^{\rm foc} \times \cdots \times M_{k_f}^{\rm foc}}_{k_f} \times \underbrace{M_1^{\rm reg} \times \cdots \times M_{k}^{\rm reg}}_{k}$$

we obviously obtain a non-degenerate singularity of type $(k_e, k_h, k_f)$ and rank $k$. The general case needs just one natural modification.

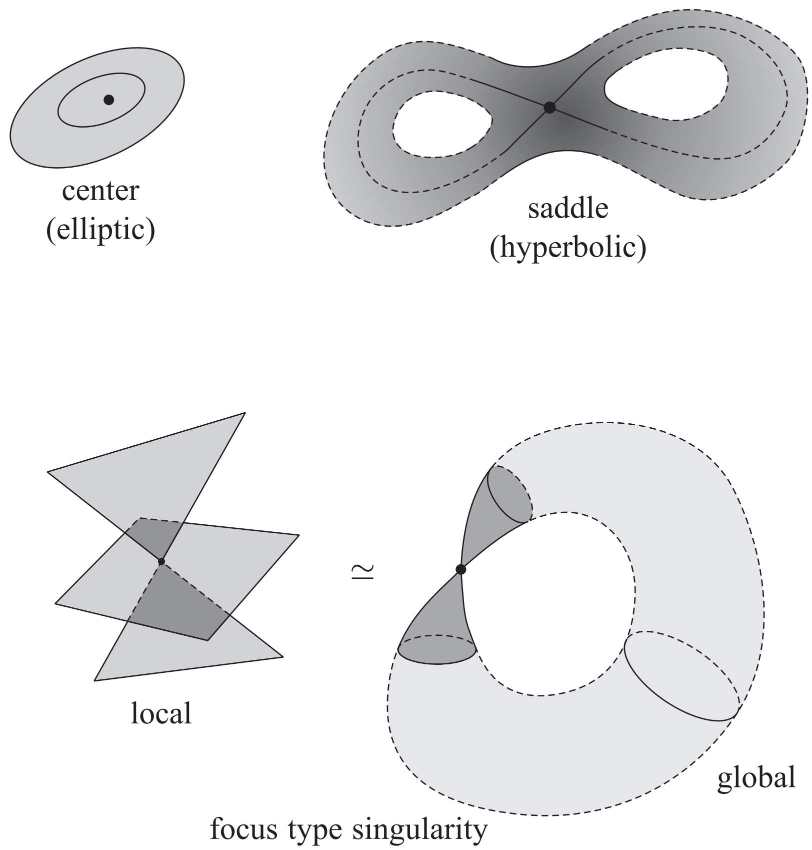


Figure 2: Basic singularities

**Theorem 6** (Nguyen Tien Zung). *Every non-degenerate singularity*[1] *is topologically equivalent to a singularity of almost direct product type*

$$\big(\underbrace{M_1^{\mathrm{ell}} \times \cdots \times M_{k_e}^{\mathrm{ell}}}_{k_e} \times \underbrace{M_1^{\mathrm{hyp}} \times \cdots \times M_{k_h}^{\mathrm{hyp}}}_{k_h} \times \underbrace{M_1^{\mathrm{foc}} \times \cdots \times M_{k_f}^{\mathrm{foc}}}_{k_f} \times M_{\mathrm{reg}}^{2k}\big) \;/\; G,$$

*where $G$ is a finite group which acts freely, symplectically, component-wise and preserves the fibration.*

Let us come back to the Neumann system (Example 1) and describe the topology of its singular fibers in terms of almost direct products. To do this, we need to introduce some notation (following A.T.Fomenko). The basic elliptic singularity $M^{\mathrm{ell}}$ shown in Fig. 2 will be denoted by $A$, the simplest hyperbolic singularity shown in Fig. 2 will be denoted by $B$, the hyperbolic singularity shown in Fig. 3

[1] In fact, one should require one additional technical condition, the so-called topological stability. We omit the definition here, see the original paper by Nguen Tien Zung and review paper by A.B. and A.Oshemkov for details.

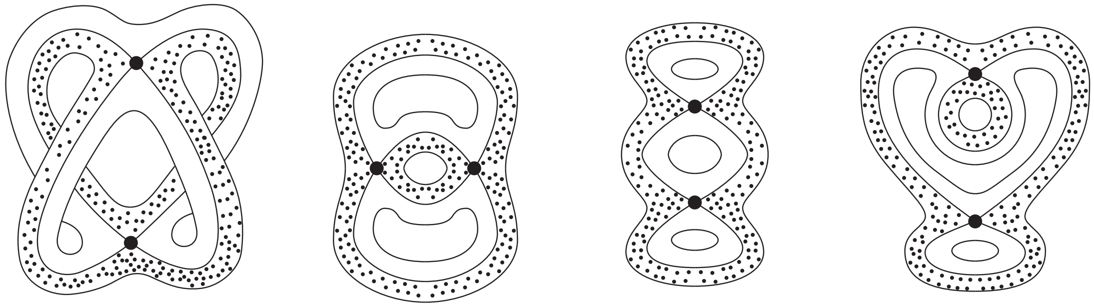

Figure 3: More examples of hyperbolic singularities

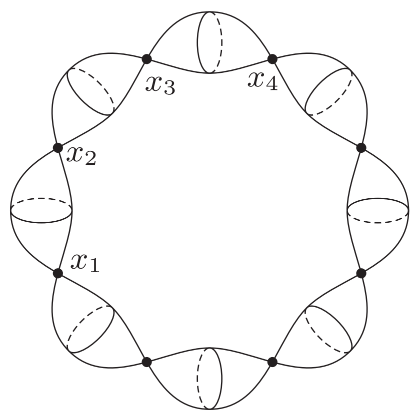


Figure 4: Focus singularity (general case)

(second in the list) will be denoted by $C_2$ and finally the regular fibration $M^{\text{reg}}$, i.e., the product $S^1 \times D^1$ will be denoted by Reg.

The topology of the singular fibers corresponding to points $P_1, P_2$ and $P_3$ is as follows (each of them is non-degenerate):

| | | |
|---|---|---|
| $P_1$ : | two copies of $A \times A$ | elliptic-elliptic type (stable) |
| $P_2$ : | $C_2 \times A$ | hyperbolic-elliptic type (unstable) |
| $P_3$ : | $C_2 \times C_2/\mathbb{Z}_2$ | hyperbolic-hyperbolic type (unstable) |

The two tangency points correspond to degenerate singularities. Their topological type is easy to describe too, they are the so-called pitchfork singularities.

Points located on the branches of the bifurcation diagram $\Sigma$ correspond to corank one singularities. Those points which lie on the boundary of the momentum map are of type $A \times \text{Reg}$ (two or four connected components), they correspond to stable periodic solutions. The points that belong to the "inner" branches of $\Sigma$ can be of two different types, either $B \times \text{Reg}$ (two copies) or $C_2 \times \text{Reg}$. The corresponding critical periodic solutions are unstable. Thus, for the Neumann system the structure of the associated Lagrangian fibration can be completely described by using standard simple models built from elementary blocks. From this analysis, we can easily obtain the "list" of stable equilibrium points and stable periodic orbits. Namely, there are two stable equilibrium points of elliptic-elliptic type. They correspond to the point $P_1$ on the bifurcation diagram $\Sigma$ (see Fig. 1). Stable periodic orbits correspond to those branches of $\Sigma$ that are located on the boundary of the image of the momentum map $\Phi$. More precisely, a periodic solution $\gamma(t)$ of the Neumann system is orbitally stable if and only if $\Phi(\gamma(t))$ belongs to the boundary of Image($\Phi$).

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# Lecture 2. Jordan-Kronecker decomposition: from Linear Algebra to bi-Poisson Geometry.

### Main ingredients

- Poisson structure: symplectic leaves, Casimir functions and singular set
- Jordan-Kronecker decomposition theorem
- Compatible Poisson structures and the family $\mathcal{F}_\Pi$ of commuting Casimirs
- Completeness criterion and codimension two principle
- From singularities of Poisson brackets to singularities of Lagrangian fibrations

## Basic Poisson geometry

We start with recalling some basic facts about Poisson structures/brackets.

**Definition 3.** A *Poisson bracket* $\{\ ,\ \}$ on a smooth manifold $M$ is a bilinear operation on the space of smooth functions $C^\infty(M)$:

$$f, g \mapsto \{f, g\}, \quad f, g, \{f, g\} \in C^\infty(M)$$

with the following properties:

1. skew symmetry: $\{f, g\} = -\{g, f\}$;
2. Leibniz rule: $\{f, gh\} = g\{f, h\} + h\{f, g\}$;
3. Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

In local coordinates, each Poisson bracket is defined by the following formula:

$$\{f, g\}(x) = \sum A^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

where $A = (A^{ij})$ is a skew-symmetric $(2, 0)$-tensor called a *Poisson structure*. (In order for this formula to define a Poisson bracket, $A$ must satisfy a system of PDEs which is equivalent to the Jacobi identity).

We do not require $A$ to be either non-degenerate or of constant rank. The rank of $A(x)$ may vary from point to point, so we define the rank of $A$ on the whole manifold $M$ to be

$$\operatorname{rank} A = \max_{x \in M} \operatorname{rank} A(x).$$

If $\operatorname{rank} A < \dim M$ then, as a rule, there exist *Casimir functions* $f \in C^\infty(M)$ such that
$$\{f, g\}_A = 0 \quad \text{for any } g \in C^\infty(M).$$

We say "as a rule", because there are examples of degenerate Poisson brackets which do not admit any global Casimir functions. However locally, in a neighbourhood of a regular point (i.e., where the rank of $A$ is locally constant) we can always find $k$ independent Casimir functions, where $k = \dim M - \operatorname{rank} A$.

Property: $f$ is Casimir $\quad\Leftrightarrow\quad df(x) \in \operatorname{Ker} A(x)$ for all $x \in M$. Moreover, for regular $x \in M$, the differentials $df(x)$ of (local) Casimirs generate $\operatorname{Ker} A(x)$.

A Poisson structure $A$ defines on $M$ a natural distribution $\mathcal{D}$ (of variable rank): at each point $x \in P$ we set $\mathcal{D}(x) = \{\xi \in T_x M \mid \xi = A\, df(x) \text{ for some } f \in C^\infty(M)\}$. In other words, $\mathcal{D}$ is generated by Hamiltonian vector fields. This distribution is integrable and $M$ can naturally be partitioned into symplectic leaves which can be characterised as integral submanifolds of $\mathcal{D}$, i.e. submanifolds $\mathcal{O} \subset M$ such that $T_x\mathcal{O} = \mathcal{D}(x)$ at each point $x \in \mathcal{O}$. The dimension of the symplectic leaf passing through $x$ equals $\operatorname{rank} A(x)$. In terms of symplectic leaves, the Casimir functions can be characterised by the property of being constant on each symplectic leaf.

To each $A$, we can assign its singular set
$$\mathsf{S}_A = \{x \in M \mid \operatorname{rank} A(x) < \operatorname{rank} A\}.$$

Equivalently, $\mathsf{S}_A$ is the union of all symplectic leaves of non-maximal dimension. One usually prefers to work with generic symplectic leaves, so the set $\mathsf{S}_A$ is often ignored, but in this course $\mathsf{S}_A$ will play a crucial role.

**Example 2.** One of the fundamental examples of Poisson brackets is the Lie-Poisson bracket associated to a finite-dimensional Lie algebra $\mathfrak{g}$. This bracket is defined on the dual space $\mathfrak{g}^*$ as follows:
$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j},$$
where $f, g \in C^\infty(\mathfrak{g}^*)$, $df(x), dg(x) \in (\mathfrak{g}^*)^* = \mathfrak{g}$ and $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

In Cartesian coordinates, the Poisson structure $A$ is linear:
$$A_{ij}(x) = \sum_k c_{ij}^k x_k,$$
and this property of *being linear* is characteristic for all Lie-Poisson brackets.

The symplectic leaves of the Lie-Poisson bracket of a Lie algebra $\mathfrak{g}$ coincide with the coadjoint orbits of the corresponding Lie group $G$. The Casimir functions are invariants of the coadjoint action of $G$ on $\mathfrak{g}^*$.

The singular set of the Lie-Poisson bracket, which we will denote by $\mathsf{Sing} \subset \mathfrak{g}^*$ in this particular case, is the union of coadjoint orbits of non-maximal dimension.

Consider three elementary examples of Lie-Poisson brackets.

**Example 3.** $so(3)$–bracket: $A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$

Casimir function: $F = x^2 + y^2 + z^2$

Symplectic leaves are spheres centred at the origin + one singular leaf $\{0\}$

Singular set is $\mathsf{S}_A = \{\operatorname{rank} A < 2\} = \{0\}$, $\operatorname{codim} \mathsf{S}_A = 3$

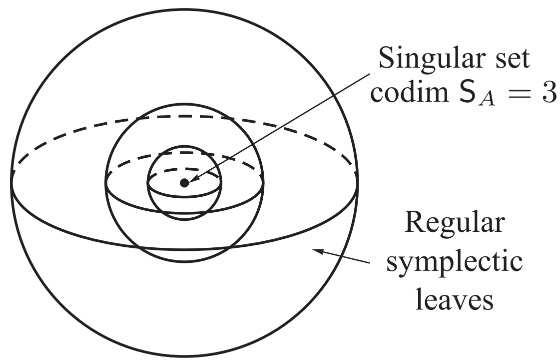


Figure 5: $so(3)$-bracket

**Example 4.** $sl(2,\mathbb{R})$–bracket: $A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix}$

Casimir function: $F = x^2 + yz$

Symplectic leaves: hyperboloids, two halves of the cone + one singular leaf $\{0\}$

Singular set is $\mathsf{S}_A = \{\operatorname{rank} A < 2\} = \{0\}$, $\operatorname{codim} \mathsf{S}_A = 3$

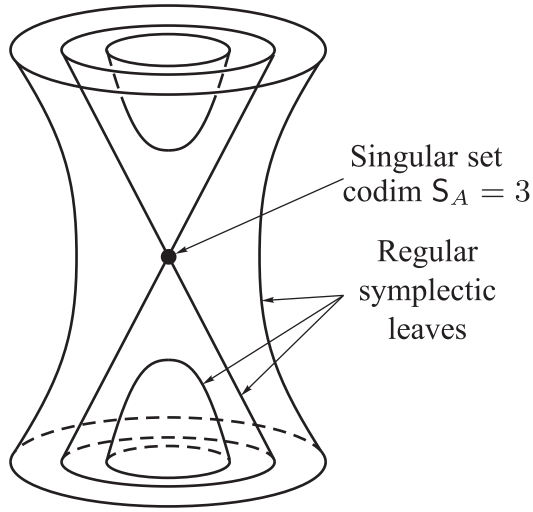


Figure 6: $sl(2)$ bracket

**Example 5.** Heisenberg–Lie bracket: $A = \begin{pmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Casimir function: $F = z$

Symplectic leaves: planes $\{z = \text{const} \neq 0\}$ + points on the plane $\{z = 0\}$

Singular set is $\mathsf{S}_A = \{\operatorname{rank} A < 2\} = \{z = 0\}$, $\operatorname{codim} \mathsf{S}_A = 1$

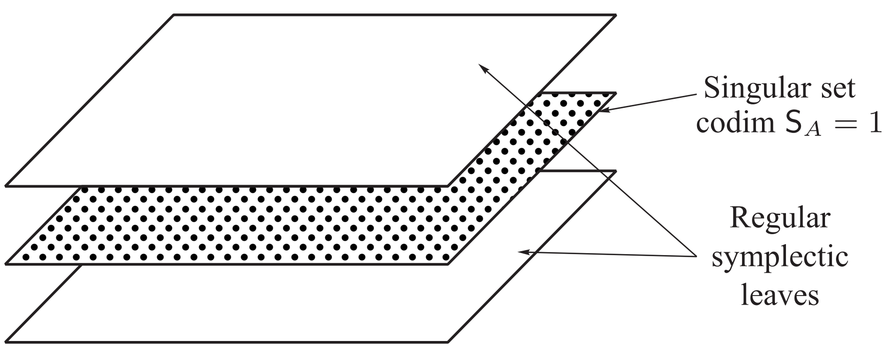


Figure 7: Heisenberg-Lie bracket

**Exercise 1.** Let $\mathfrak{g} = gl(n, \mathbb{R})$ and identify $gl(n, \mathbb{R})^*$ with $gl(n, \mathbb{R})$ by using the invariant inner product $\langle A, B \rangle = \operatorname{tr} AB$. Prove that $C$ is a singular element (i.e. $C \in \mathsf{Sing}$) if and only if there exists an eigenvalue $\lambda \in \mathbb{C}$ of $C$ admitting two or more linearly independent eigenvectors. Equivalently, we can reformulate this condition by saying that the minimal polynomial of $C$ coincides with the characteristic polynomial $\chi(t) = \det(C - t \cdot \mathrm{Id})$.

## Jordan–Kronecker decomposition

Let $A$ and $B$ be two Poisson structures on a manifold $M$. They are said to be compatible if their sum $A+B$ is a Poisson structure too. Before studying properties of compatible Poisson brackets we want to discuss a purely algebraic question. At a fixed point $x \in M$, we can think of $A$ and $B$ as a pair of skew symmetric bilinear forms defined on the cotangent space $V = T_x^*M$. What is a canonical form of such a pair on $V$?

The following theorem gives the complete answer to this question.

**Theorem 7.** *Let $A$ and $B$ be two skew-symmetric bilinear forms on a (complex) vector space $V$. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:*

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \qquad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

*where the pairs of the corresponding blocks $A_i$ and $B_i$ can be of the following three types:*

| | $A$ | $B$ |
|---|---|---|
| *Jordan block* ($\lambda \in \mathbb{C}$) | $\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$ | $\begin{pmatrix} & -\mathrm{Id} \\ \mathrm{Id} & \end{pmatrix}$ |
| *Jordan block* ($\lambda = \infty$) | $\begin{pmatrix} & \mathrm{Id} \\ -\mathrm{Id} & \end{pmatrix}$ | $\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$ |
| *Kronecker block* | $\left(\begin{array}{cc} & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & \end{array}\right)$ | $\left(\begin{array}{cc} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & \end{array}\right)$ |

*where $J(\lambda)$ denotes a usual Jordan $\lambda$-block:* $J(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$.

This theorem immediately implies several very interesting and useful corollaries. Before discussing them, we introduce some natural terminology. Along with $A$ and $B$, it is convenient to consider the pencil of skew symmetric forms generated by them, i.e. all linear combinations $\mu A + \lambda B$. To kill one parameter, we are going to consider the forms up to proportionality, so we define the pencil of forms as $\Pi = \{A_\lambda = A + \lambda B\}$, where $\lambda$ is a complex number or $\infty$ (we set by definition $A_\infty = B$). In other words, we consider the "projective version" of the pencil so that its parameter $\lambda$ runs over the projective space $\mathbb{C}P^1 = \overline{\mathbb{C}}$.

The numbers $\lambda$'s that appear in this theorem as eigenvalues of Jordan blocks $J(\lambda)$ are exactly those for which the rank of $A_\lambda = A + \lambda B$ drops. More precisely, we define the rank of the pencil to be $\operatorname{rank} \Pi = \max_{\lambda \in \overline{\mathbb{C}}} \operatorname{rank} A_\lambda$. We say that $A_\mu$ is generic if $\operatorname{rank} A_\mu = \operatorname{rank} \Pi$. The set of singular values $\lambda_i$ of the parameter $\lambda$, i.e. such that $\operatorname{rank} A_{\lambda_i} < \operatorname{rank} \Pi$, will be called the *spectrum* of the pencil:

$$\Lambda = \{\lambda_1, \dots, \lambda_k\}.$$

Notice that $\Lambda$ might be empty. The elements of $\Lambda$ are exactly the diagonal elements of the Jordan blocks that appear in the Jordan-Kronecker decomposition.

As the first corollary of the Jordan–Kronecker theorem, we obtain the following important fact:

| for every pair of skew symmetric forms there exists a bi-Lagrangian subspace |
|---|

,

i.e. a subspace $U \subset V$ which is isotropic w.r.t. both $A$ and $B$ (as well as any linear combination $A + \lambda B$) and, moreover, is maximal isotropic w.r.t. all generic combinations $A + \mu B$. In particular, $\dim U = \frac{1}{2}\big(\dim V + \operatorname{corank} \Pi\big)$. Indeed, to construct such a subspace it is sufficient to consider the direct sum of the subspaces corresponding to the right lower zero sub-blocks of each block in the Jordan-Kronecker decomposition.

This fact is an algebraic counterpart of the following principle in the theory of bi-Hamiltonian system:

| Bi-Hamiltonian systems are, as a rule, integrable and, moreover, they admit a complete family of first integrals in bi-involution. |
|---|

Sometimes, such a bi-Lagrangian subspace admits a natural invariant description. Indeed, consider the simplest case of a single Kronecker block:

$$A - \lambda B = \begin{pmatrix} & \boxed{\begin{matrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{matrix}} \\ \boxed{\begin{matrix} -1 & & \\ \lambda & \ddots & \\ & \ddots & -1 \\ & & \lambda \end{matrix}} & \end{pmatrix}$$

The kernel of $A - \lambda B$ is generated by $(0, \dots, 0, \lambda^k, \lambda^{k-1}, \dots, \lambda, 1)$. This leads us to the following observation.

**Proposition 1.** *Let* $L = \mathrm{Span}\{\mathrm{Ker}(A - \lambda B)\}_{\lambda \in \mathbb{R}}$. *Then*

- *$L$ is isotropic w.r.t. every form $A - \lambda B$;*
- *$L$ is maximal (!) isotropic, i.e. bi-Lagrangian.*

This observation can easily be generalised to the case of many blocks. Consider the subspace $L = \mathrm{Span}\{\mathrm{Ker}(A+\mu B)\}_{\mu \notin \Lambda}$ generated by the kernels of generic forms. It is easy to see that each Kronecker block $A_i + \mu B_i$ gives a non-trivial contribution to $L$ (see Proposition 1), whereas Jordan blocks $A_j + \mu B_j$ do not contribute at all, since for generic $\mu$ they are non-degenerate and $\mathrm{Ker}(A_j + \mu B_j) = \{0\}$.

Proposition 1 implies that $L$ is bi-isotropic and the next theorem gives necessary and sufficient conditions for $L$ to be bi-Lagrangian.

**Theorem 8.**

1. *The subspace $L = \mathrm{Span}\{\mathrm{Ker}(A + \mu B)\}_{\mu \notin \Lambda}$ is bi-isotropic.*
2. *The following conditions are equivalent:*
    - *$L$ is bi-Lagrangian,*
    - *the pencil $\Pi$ is of pure Kronecker type, i.e. its Jordan-Kronecker decomposition consists of Kronecker blocks only,*
    - *the spectrum $\Lambda$ of the pencil $\Pi$ is empty,*
    - *a bi-Lagrangian subspace is unique.*

**Exercise 2.** Derive this theorem from the Jordan-Kronecker decomposition.

## Dictionary

Our next goal is to pass from Linear Algebra to bi-Poisson Geometry. Here is a kind of dictionary that helps to translate ideas and constructions from Algebra to Geometry.

| | | |
|---|---|---|
| skew-symmetric form | $\longleftrightarrow$ | Poisson structure |
| isotropic subspace | $\longleftrightarrow$ | commuting functions |
| pencil of skew-symmetric forms | $\longleftrightarrow$ | compatible Poisson structures |
| kernel of a skew-symmetric form | $\longleftrightarrow$ | Casimir functions |
| maximal isotropic subspace | $\longleftrightarrow$ | integrable system |
| bi-Lagrangian subspace | $\longleftrightarrow$ | functions in bi-involution |

Everything we are going to discuss below can be considered as a local differential geometry of compatible Poisson brackets. In what follows, we will use the following general idea. Every geometric object in local coordinates can be expanded into power series. Then we can analyse this expansion up to a certain order. In this view, the material of this lecture can be understood as zero-order analysis of compatible Poisson brackets. Namely, we are going to analyse the "values" of Poisson brackets at some point, but not the "derivatives". Surprisingly, even such a naive analysis will lead us to rather non-trivial conclusions. The next lecture is devoted to the first order analysis of compatible brackets.

## Compatible Poisson brackets and commuting Casimirs

**Definition 4.** Two Poisson structures $A$ and $B$ are said to be *compatible* if $A + B$ is again a Poisson structure.

**Exercise 3.** For compatibility of $A$ and $B$ we only need to check the Jacobi identity for $A + B$ which, in local coordinates, amounts to a system of PDEs. Write down these equations explicitly. Check that the compatibility of $A$ and $B$ implies that every linear combination $\mu A + \lambda B$ is a Poisson structure too.

Let $M$ be a manifold endowed with a linear family $\Pi = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. First of all we notice that such pencils may be of different algebraic types in the sense of the Jordan-Kronecker theorem. By the algebraic type of $\Pi$, we mean the type of JK decomposition for the pair $A(x)$ and $B(x)$ at a point $x \in M$ (i.e. the number and sizes of Jordan and Kronecker blocks). The problem, however, is that this type may vary from point to point. On the other hand, the number of different algebraic types is finite, and therefore there must be an open everywhere dense subset of points $x \in M$ where this type is locally constant. We call such points *generic* and assume that all of them have the same algebraic type [2]. The *algebraic type of the pencil* $\Pi$ of compatible Poisson brackets

[2] In all reasonable examples this assumption is fulfilled. But in general one can construct "artificial" examples where $M$ contains several open domains related to different algebraic types of the Poisson pencil.

is, by definition, the type of $\Pi(x)$ at a generic point $x \in M$.

Thus, basically there are three essentially different types of Poisson pencils:

- Kronecker (only Kronecker blocks),
- symplectic (only Jordan blocks; this is equivalent to the fact that $A_\lambda(x)$ is *typically* non-degenerate),
- mixed (both Kronecker and Jordan blocks).

Compatible Poisson brackets of Kronecker and symplectic types have been extensively studied in many papers (see some references below).

Assume that all $A_\lambda \in \Pi$ are degenerate so that each of them possesses non-trivial Casimir functions. We will say that $\mu \in \mathbb{R}$ is generic if $\operatorname{rank} A_\mu$ is maximal in $\Pi$. To avoid some unpleasant situations, we will always assume that the Poisson structures we are dealing with are all real-analytic so that $\operatorname{rank} A_\mu$ is constant on an open dense subset.

The following proposition describes one of the mechanisms of integrability of bi-Hamiltonian systems and is well-known in Poisson geometry.

**Proposition 2.** *Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \Pi$. Then*

*1) the family of functions*

$$\mathcal{F}_\Pi = \{\textit{all Casimir functions of all generic brackets } A_\mu\}$$

*consists of first integrals of $v(x)$;*

*2) these integrals commute (w.r.t. every $A_\lambda \in \Pi$).*

*Proof.* The first part of Proposition 2 is obvious. Indeed, if $v(x)$ is Hamiltonian w.r.t. a Poisson structure $A$, then the Casimirs of $A$ are first integrals of $v$. The second part follows immediately from item 1 of Theorem 8 (hint: use the above dictionary). □

**Example 6** (Argument shift method)**.**

On the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$ there are two natural compatible Poisson brackets:

$$\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad \text{and} \quad \{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j},$$

where $a = (a_i) \in \mathfrak{g}^*$ is a fixed element.

**Proposition 3.** *For each $\lambda \in \mathbb{R}$, the bracket $\{\ ,\ \}_\lambda = \{\ ,\ \} + \lambda\{\ ,\ \}_a$ is isomorphic to $\{\ ,\ \}$ (by means of translation $x \to x + \lambda a$). In particular,*

- *the Casimir functions of* $\{\ ,\ \}_\lambda$ *are of the form* $f(x+\lambda a)$*, where* $f$ *is a coadjoint invariant of* $\mathfrak{g}$*;*
- *the singular set of* $\{\ ,\ \}_\lambda$ *is* $\mathsf{Sing} + \lambda a$*, where* $\mathsf{Sing} \subset \mathfrak{g}^*$ *is the set of singular coadjoint orbits of* $\mathfrak{g}$*;*
- *the kernel of* $\{\ ,\ \}_\lambda$ *at the point* $x \in \mathfrak{g}^*$ *is the* $\mathrm{ad}^*$*-stationary subalgebra of* $x + \lambda a$*, i.e.,* $\mathrm{Ann}\,(x+\lambda a) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi(x+\lambda a) = 0\}$*.*

For this special kind of a Poisson pencil $\{\ ,\ \}_\lambda$ on $\mathfrak{g}^*$ we can use Proposition 3 to construct the family of commuting functions

$$\{f(x+\lambda a) \mid \lambda \in \mathbb{R},\ f \text{ is a coadjoint invariant of } \mathfrak{g}\}.$$

To avoid possible issues related to non-existence of global Casimirs, we modify this construction by replacing the functions $f(x+\lambda a)$ with the homogeneous polynomials obtained by expansion of local Casimirs into Taylor series:

$$f(a+\lambda x) = f(a) + \lambda f_1(x) + \lambda^2 f_2(x) + \cdots + \lambda^k f_k(x) + \ldots$$

at the chosen point $a \in \mathfrak{g}^*$ (assuming that $a$ is regular so that local Casimirs exist and their differentials at $a$ generates $\mathrm{Ann}\, a \subset \mathfrak{g}$). Thus, we set

$$\mathcal{F}_a = \{f_k(x) \mid f \text{ is a local Casimir at } a \in \mathfrak{g}^*, k \in \mathbb{N}\}$$

and call $\mathcal{F}_a$ so obtained the *family of polynomial shifts*. All functions of $\mathcal{F}_a$ are in bi-involution w.r.t. the brackets $\{\ ,\ \}$ and $\{\ ,\ \}_a$.

Let us repeat the conclusion of Proposition 2 once again: if we have a dynamical system which is Hamiltonian w.r.t. a pencil $\Pi$ of degenerate Poisson brackets [3], then we immediately obtain a family $\mathcal{F}_\Pi$ of commuting integrals by taking the Casimirs of all (generic) brackets $A_\mu \in \Pi$.

For many examples of Hamiltonian systems in geometry, classical mechanics and mathematical physics (such as the Toda lattice, integrable geodesic flows on Lie groups and homogeneous spaces, integrable cases in rigid body dynamics, etc.), this family $\mathcal{F}_\Pi$ of commuting integrals guarantees the Liouville integrability and, in particular, defines the structure of a singular Lagrangian fibration (into invariant tori) on the phase space $M$.

So it is a natural question to ask: what are the properties of $\mathcal{F}_\Pi$? Namely:

1. Completeness

[3] If the pencil $\Pi = A + \lambda B$ is symplectic, i.e. a typical bracket, say $B$, is non-degenerate, then one uses another method of producing commuting integrals. To get integrals of a bi-Hamiltonian system in this case, it is sufficient to consider the eigenvalues of the recursion operator $R = AB^{-1}$, or equivalently, the traces $\mathrm{tr}\, R^k$, $k = 1, 2, \ldots$

2. Set of critical points
3. Equilibrium points
4. Non-degeneracy conditions, types
5. Codimension one singularities
6. Stability

Is there any relationship between these properties and the properties of the corresponding family $\Pi$ of compatible Poisson brackets? Obviously, YES!

Family of compatible Poisson brackets
↓
Commuting integrals
↓
Lagrangian fibration
↓
Its singularities
↓
Dynamical properties and stability analysis

But how close is this relation? A surprising fact is that this relationship

properties of singularities of $\mathcal{F}_\Pi$ $\leftrightarrow$ properties of $\Pi$

is very close and much more straightforward than one could expect.

## Properties of $\mathcal{F}_\Pi$

"Zero order" analysis allows us to answer questions 1, 2 and 3. To that end, we first need to make one important remark. The above construction of commuting Casimirs, i.e. family $\mathcal{F}_\Pi$, is based on the assumption that Casimir functions do exist. Recall that it is not always the case: there are degenerate Poisson brackets which do not admit any non-trivial Casimir functions (the foliation into symplectic leaves may, for example, be chaotic). Thus, we need either to assume that global Casimir functions exist for all Poisson structures from the pencil, or to work with local Casimir functions which always exist near a generic point. The second option is simpler and we prefer to work in a small neighbourhood $U$ of a point $x \in M$ under the following assumptions:

1) almost all Poisson structures $A_\lambda \in \Pi$ have maximal rank at $x$, i.e. $\operatorname{rank} A_\lambda(x) = \operatorname{rank} \Pi$;

2) for definiteness, we assume that $\operatorname{rank} A_0(x) = \operatorname{rank} \Pi$, so that the symplectic leaf $\mathcal{O}(x)$ of $A_0$ passing through $x$ has maximal possible dimension;

3) Casimir functions of $A_0$ are defined on $U$ and the differentials of these Casimirs at $x$ span $\operatorname{Ker} A_0(x) \subset T^*_x M$;

4) the same condition holds for $A_\varepsilon$ for sufficiently small $\varepsilon \in \mathbb{R}$, $\varepsilon \in (-\delta, \delta)$;

5) the family $\mathcal{F}_\Pi$ consists of (local) Casimir functions of Poisson structures $A_\varepsilon$, $\varepsilon \in (-\delta, \delta)$.

The latter excludes the Casimir functions of those brackets $A_\lambda$ whose rank drops at the point $x_0$. This technical condition allows us to avoid the discussion about the behaviour of Casimir functions at singular points which could be quite unpredictable.

**Completeness.**

First of all we need to find out if the family $\mathcal{F}_\Pi$ is complete, i.e., sufficient to guarantee complete integrability?

The completeness means that $\mathcal{F}$ contains

$$s = \frac{1}{2}(\dim M + \operatorname{corank} \Pi)$$

independent functions. Instead of computing the number of independent integrals in $\mathcal{F}_\Pi$ it is much better to use the following definition: $\mathcal{F}_\Pi$ is *complete* if at a generic point $x \in M$ the differentials $df(x)$, $f \in \mathcal{F}_\Pi$, generate a maximal isotropic subspace.

**Theorem 9.** *The family $\mathcal{F}_\Pi$ is complete if and only if at a generic point $x \in M$ the following condition holds:*

$$\operatorname{rank} A_\lambda(x) = \operatorname{rank} \Pi \quad \textit{for all } \lambda \in \overline{\mathbb{C}},$$

*or equivalently, the pencil $\Pi$ is of Kronecker type.*

*Proof.* Use Theorem 8 and the *Linear Algebra* $\leftrightarrow$ *Poisson Geometry* dictionary. □

The completeness condition means that all the brackets $A_\lambda \in \Pi$ must be of the same rank, but not only this. The rank of each particular bracket $A_\lambda$ may drop at some points, in other words the singular set $\mathsf{S}_\lambda = \mathsf{S}_{A_\lambda}$ may be non-trivial, but there must be points which do not belong to any of these singular sets $\mathsf{S}_\lambda$, $\lambda \in \overline{\mathbb{C}}$. This observation leads us to the following

**Codimension two principle.** Let all the brackets $A_\lambda$, $\lambda \in \overline{\mathbb{C}}$ have the same rank and $\operatorname{codim} \mathsf{S}_\lambda \geq 2$ for almost all $\lambda$'s. Then $\mathcal{F}_\Pi$ is complete.

**Set of critical points**.

Suppose that for a pencil $\Pi$ the completeness condition holds and, therefore, the family $\mathcal{F}_\Pi$ of commuting Casimirs is complete and defines the structure of the Lagrangian fibration on $M$. However, there are still some singular points $x \in M$ where the commuting functions from $\mathcal{F}_\Pi$ become dependent:

$$\mathsf{S}_\Pi = \{x \in M \mid \dim D_{\mathcal{F}_\Pi}(x) < \frac{1}{2}(\dim M + \operatorname{corank} \Pi)\}$$

where $D_{\mathcal{F}_\Pi}(x) \subset T^*_x M$ is the subspace spanned by the differentials of $f \in \mathcal{F}_\Pi$.

$\mathsf{S}_\Pi$ is, by definition, the set of critical points of $\mathcal{F}_\Pi$ (or, equivalently the singular set of the corresponding Lagrangian fibration (see Lecture 1)).

On the other hand, for every $\lambda \in \overline{\mathbb{C}}$, we can define the set of "singular points" of $A_\lambda$ in $M$:

$$\mathsf{S}_\lambda = \{x \in M \mid \operatorname{rank} A_\lambda(x) < \operatorname{rank} \Pi\}.$$

Is there any relationship between these singular sets $\mathsf{S}_\Pi$ and $\mathsf{S}_\lambda$, i.e. the singular set of the Lagrangian fibration and the singular sets of Poisson structures $A_\lambda \in \Pi$? The answer is very natural.

**Theorem 10.** *A point $x$ is critical for $\mathcal{F}_\Pi$ iff there is $\lambda \in \overline{\mathbb{C}}$ such that $x \in \mathsf{S}_\lambda$. In other words, the set of critical points $\mathsf{S}_\Pi$ of the family $\mathcal{F}_\Pi$ is the union of "singular sets" $\mathsf{S}_\lambda$ over all $\lambda \in \overline{\mathbb{C}}$:*

$$\mathsf{S}_\Pi = \bigcup_{\lambda \in \overline{\mathbb{C}}} \mathsf{S}_\lambda$$

*Proof.* Use Theorem 8 and the Linear Algebra $\leftrightarrow$ Poisson Geometry dictionary. $\square$

**Common equilibria.**

Let $\mathcal{F}$ be a family of commuting functions on a Poisson manifold $M$. We say that $x \in M$ is a *common equilibrium* point for $\mathcal{F}$ if $X_f(x) = 0$ for all $f \in \mathcal{F}$.

Why are we interested in *common* equilibria? What is the difference between common equilibria and equilibria for a particular Hamiltonian function $H \in \mathcal{F}$? Assume that $x_0$ is an isolated equilibrium point for $H$ on a symplectic manifold (or on a symplectic leaf in the Poisson case). Then for any function $f$ commuting with $H$ (i.e. for any first integral of the Hamiltonian vector field $X_H$), this point $x_0$ is an equilibrium too. In other words, an isolated equilibrium point is a common equilibrium for all integrals of the Hamiltonian system. If $x_0$ is not *common*, then the orbit of the Hamiltonian action generated by the integrals $f \in \mathcal{F}$ is nontrivial and this orbit consists entirely of equilibrium points of $X_H$. It can be shown that in the latter case, these equilibrium points cannot be stable (provided the integrable system is not resonant). Thus, common equilibria are typical and very important for stability analysis.

**Theorem 11.** *A point $x \in M$ is a common equilibrium for $\mathcal{F}_\Pi$ if and only if the kernels of all generic brackets at this point coincide:*

$$\operatorname{Ker} A_\lambda(x) = \operatorname{Ker} A_\mu(x),$$

*for all generic $\lambda$ and $\mu$.*

*Proof.* This is another statement from Linear Algebra (do not forget about the dictionary). Indeed, $x$ is a common equilibrium point if $A_0\big(D_{\mathcal{F}_\Pi}(x)\big) = 0$[4]. Thus, $D_{\mathcal{F}_\Pi} = \sum_{\mu\notin\Lambda(x)} \operatorname{Ker} A_\mu(x) = \operatorname{Ker} A_0$ and, consequently, $\operatorname{Ker} A_0(x) = \operatorname{Ker} A_\mu(x)$, $\mu \notin \Lambda(x)$, as required. □

Let us apply these results to the argument shift method. For an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ we consider the family of polynomial shifts $\mathcal{F}_a$ on $\mathfrak{g}^*$ (see example 6). When is this family complete? What is the structure of the singular set of $\mathcal{F}_a$? What are common equilibria? To answer these questions we only need to reformulate the general results for this specific pencil.

**Theorem 12.** *1. The family $\mathcal{F}_a$ of polynomial shifts is complete if and only if*

$$\operatorname{codim} \mathsf{Sing} \geq 2,$$

*where $\mathsf{Sing} \subset \big(\mathfrak{g}^{\mathbb{C}}\big)^*$ is the singular set for the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$.*

*2. The singular set $\mathsf{S}_a = \big\{x \in \mathfrak{g}^* \mid \dim \operatorname{span}(dh(x), h \in \mathcal{F}_a) < \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})\big\}$ is the "a"-cylinder over the singular set $\mathsf{Sing}$. More precisely,*

$$x \in \mathsf{S}_a \quad \Leftrightarrow \quad x = y + \lambda a \ \text{ for some } y \in \mathsf{Sing} \subset \big(\mathfrak{g}^{\mathbb{C}}\big)^* \text{ and } \lambda \in \mathbb{C}$$

*3. A regular point $x \in \mathfrak{g}^*$ is a common equilibrium for $\mathcal{F}_a$ if and only if*

$$\operatorname{Ann} a = \operatorname{Ann} x.$$

If $\mathfrak{g}$ is semisimple, then $\operatorname{codim} \mathsf{Sing} = 3$ and we get the famous Fomenko-Mischenko theorem about completeness of shifts in the semisimple case. The common equilibria $x \in \mathfrak{g}^* = \mathfrak{g}$ are easy to describe too. These are exactly the elements of the Cartan subalgebra generated by $a$. In particular, the number of common equilibria on a generic (co)adjoint orbit is equal to the order of the Weil group (which is the same as the number of intersection points of such an orbit with the Cartan subalgebra).

**Remark 2.** We emphasise again that all these results (about completeness, singular set and common equilibria) are of "zero order" in the sense that they only require the information about the forms $A(x)$ and $B(x)$ at a fixed point $x \in M$, but not about their derivatives!

From *Differential* Geometry we only need one simple thing: at a generic point $x \in M$ the differential of Casimir functions generate the kernel of the bracket. After this, everything else is just a simple corollary of the Jordan-Kronecker decomposition theorem.

[4] Here we consider $A_0$ as a "distinguished" Poisson bracket to compute Hamiltonian vector fields. However, $A_0$ can be replaced by any other regular bracket from the pencil.

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# Lecture 3. Linearisation of Poisson pencils and a criterion of non-degeneracy

**Main ingredients:**

1. Linearisation of a Poisson pencil
2. Linear pencils
3. Classification of non-degenerate linear pencil
4. General non-degeneracy criterion

## Linearisation of a Poisson structure

According to the Weinstein splitting theorem, each Poisson structure $A$ locally splits into direct product of a non-degenerate Poisson structure $A_{\mathrm{sympl}}$ and the transversal structure $A_{\mathrm{transv}}$ which vanishes at the given point:

$$A = A_{\mathrm{sympl}} \times A_{\mathrm{transv}}.$$

In terms of local coordinates, this means that in a neighbourhood of a point $P_0 \in M$ there exists a coordinate system $p_1, \dots, p_k, q_1, \dots, q_k, x_1, \dots, x_m$, $2k + m = \dim M$ in which the Poisson structure $A$ takes the following form:

$$A = \begin{pmatrix} \boxed{\begin{matrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & \omega_{12}(x) & \dots & \omega_{1m}(x) \\ -\omega_{12}(x) & 0 & & \vdots \\ & & \ddots & \\ \vdots & & 0 & \omega_{m-1,m}(x) \\ -\omega_{1m}(x) & \dots & -\omega_{m-1,m}(x) & 0 \end{matrix}} \end{pmatrix}.$$

The $\omega$-block of this matrix represents a Poisson structure in variables $x$ (called the *transversal Poisson structure*) and at the point $P_0$ all the entries of the $\omega$-block vanish.

The transversal Poisson structure $A_{\rm transv}$ is well defined and we can consider its linearisation just by taking the linear terms in the Taylor expansion

$$A_{\rm transv}(x) = \left(\sum_k c_{ij}^k x_k + \dots\right) \quad \mapsto \quad A_{\rm linear}(x) = \left(\sum_k c_{ij}^k x_k\right)$$

**Definition 5.** From the algebraic viewpoint, the *linearisation* of $A$ (more precisely of its transversal part $A_{\rm transv}$) at a point $P_0 \in M$ is a Lie algebra $\mathfrak{g}_A$ defined on $\operatorname{Ker} A(P_0) \subset T^*_{P_0}M$ as follows. Let $\xi, \eta \in \operatorname{Ker} A(P_0)$ and $f, g$ be smooth functions such that $df(x) = \xi$, $dg(x) = \eta$. Then, by definition,

$$[\xi, \eta] = d\{f, g\}(P_0) \in \operatorname{Ker} A(P_0).$$

**Remark 3.** If $P_0 \in M$ is a regular point, then $\mathfrak{g}_A$ is obviously trivial. In other words, the linearisation of $A$ becomes non-trivial only on the singular set $\mathsf{S}_A$.

**Example 7.** . Let $A$ be a Lie-Poisson bracket associated with a Lie algebra $\mathfrak{g}$. What is the linearisation of $A$ at a point $a \in \mathsf{Sing} \subset \mathfrak{g}^*$? The answer is very natural. The linearisation of $A$ at $a \in \mathfrak{g}^*$, as a Lie algebra, is the annihilator of $a$, i.e. the stationary subalgebra of $a$ in the sense of the coadjoint representation:

$$\operatorname{Ann} a = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi a = 0\}.$$

**Exercise 4.** Prove this.

## Linearisation of a Poisson pencil

Now consider $\Pi = \{A_\lambda = A + \lambda B\}$, a pencil of compatible Poisson brackets on $M$. For simplicity, assume that $B$ has maximal rank at $x$. Let us take $x \in M$, fix $\lambda \neq \infty$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$, we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, i.e. the linearisation of $A_\lambda$ at the point $x$,
- the restriction of $B$ onto $\operatorname{Ker} A_\lambda(x)$.

We can think of them as two Poisson structures on $\mathfrak{g}^*_\lambda$:

- the first one is linear, i.e., the standard Lie-Poisson structure related to $\mathfrak{g}_\lambda$,
- the second one is constant $B|_{\mathfrak{g}_\lambda}$.

**Proposition 4.** *These two Poisson structures are compatible, i.e. define a Poisson pencil* $\mathrm{d}_\lambda\Pi(x)$.

**Exercise 5.** Prove this.

**Definition 6.** *This Poisson pencil* $\mathrm{d}_\lambda \Pi(x)$ *is called the* $\lambda$*-linearisation of the pencil* $\Pi$ *at* $x \in M$.

It is easy to see that this construction is non-trivial only for $\lambda$ from the spectrum $\Lambda(x)$ of the pencil $\Pi(x) = \{A(x) + \lambda B(x)\}$ of skew-symmetric 2-forms, otherwise both structures $\mathfrak{g}_\lambda$ and $B|_{\mathfrak{g}_\lambda}$ vanish.

How to find the $\lambda$-linearisation in practice? For simplicity, assume $\lambda = 0$. Choose a coordinate system $x_1, \dots, x_k, x_{k+1}, \dots x_n$ such that

$$A(x) = \begin{pmatrix} A_1(x) & A_2(x) \\ -A_2^\top(x) & A_3(x) \end{pmatrix}, \qquad B(x) = \begin{pmatrix} B_1(x) & B_2(x) \\ -B_2^\top(x) & B_3(x) \end{pmatrix}$$

where $A_1(0) = 0$, $A_2(0) = 0$ and $A_3(0)$ is non-degenerate (in other words, the first coordinates $x_1, \dots, x_k$ "generate" the kernel of $A$ at $x = 0$).

Then the linear terms of $A_1(x)$ do not depend on $x_{k+1}, \dots, x_n$ and form a linear Poisson bracket (check this!). The constant bracket is simply $B_1(0)$.

Thus, the linearisation of this pencil at $x = 0$ is defined by the linear part of $A_1$ and the constant part of $B_1$.

In practice (see the next lecture), such a coordinate system can often be found explicitly.

**Example 8** (Argument shift method again)**.**

Consider the standard "shift argument" pencil $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ corresponding to a Lie algebra $\mathfrak{g}$ and $a \in \mathfrak{g}^*$ (see Example 6). Let $\operatorname{codim} \mathsf{Sing} \geq 2$ so that the argument shift method gives a complete family $\mathcal{F}_a$ of commuting polynomials.

Let $x \in \mathfrak{g}^*$ be a singular point for $\mathcal{F}_a$. This means that $x + \lambda a$ is a singular element of $\mathfrak{g}^*$ for some $\lambda \in \mathbb{C}$ (see Theorem 12). What is the $\lambda$-linearization $\mathrm{d}_\lambda \Pi(x)$ of this pencil at this point?

The answer is very natural. Namely, $\mathfrak{g}_\lambda$ is the $\mathrm{ad}^*$-stationary subalgebra of $x + \lambda a \in \mathfrak{g}^*$:

$$\mathfrak{g}_\lambda = \mathrm{Ann}\,(x + \lambda a) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi (x + \lambda a) = 0\}$$

and the constant bracket on $\mathfrak{g}^*_\lambda$ is $\{\ ,\ \}_{\mathrm{pr}(a)}$ where $\mathrm{pr}(a) \in \mathrm{Ann}\,(x+\lambda a)^*$ is the natural projection of $a$ from $\mathfrak{g}^*$ to $\mathrm{Ann}\,(x+\lambda a)^*$ induced by the inclusion $\mathrm{Ann}\,(x+\lambda a) \subset \mathfrak{g}$.

## Linear pencils

Thus, we see that the linearisation of an arbitrary Poisson pencil is again a pencil of compatible Poisson brackets but of a much simpler nature. Namely, one bracket is linear and the other is constant. Let us discuss some basic properties of such pencils (we shall call them *linear*).

Consider two compatible Poisson brackets on a vector space $V$:

linear $A$ + constant $B$.

What is the "compatibility condition" for these kind of brackets?

We have already discussed the most important example of such a situation. This is the "argument shift pencil" defined on the dual space of a finite-dimensional Lie algebra $\mathfrak{g}$. The brackets

$$\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad \text{and} \quad \{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

are compatible for each $a = (a_i) \in V = \mathfrak{g}^*$.

The situation can, however, be different. Namely, for the Lie-Poisson bracket $\{f,g\}_A(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ there may exist constant compatible brackets

$$\{f,g\}_B(x) = \sum B_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

which are not of the above type. The compatibility condition can be written as

$$B([\xi,\eta],\zeta) + B([\eta,\zeta],\xi) + B([\zeta,\xi],\eta) = 0.$$

This identity has a natural cohomological interpretation, namely, $B$ is a 2-cocycle in terms of the Chevalley-Eilenberg complex. The bracket $\{\ ,\ \}_a$, in this sense, is a coboundary.

For some classes of Lie algebras (e.g., semisimple), every 2-cocycle is a coboundary so that the "argument shift pencil" is the only possible linear pencil for $\mathfrak{g}$.

Thus, the linear pencils can be understood as a generalisation of the "shift of argument" construction. From the algebraic viewpoint, a linear pencil $\Pi = \{A+\lambda B\}$ is a pair $\mathfrak{g}$, $B$ where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $B$ is a 2-cocycle on $\mathfrak{g}$. We shall denote such pencils by $\Pi^{\mathfrak{g},B}$.

## Non-degenerate linear pencils

For this special kind of Poisson pencils $\Pi = \Pi^{\mathfrak{g},B}$ we can construct the family of commuting functions $\mathcal{F}_\Pi$ and ask ourselves about the structure of its singular points. We will say that $\Pi^{\mathfrak{g},B}$ is *complete*, if $\mathcal{F}_\Pi$ is complete.

**Definition 7.** *We say that a complete linear pencil $\Pi^{\mathfrak{g},B}$ is non-degenerate, if $0 \in \mathfrak{g}^*$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$.*

**Example 9** (Semisimple case: $so(3)$)**.** Let $\mathfrak{g} \simeq so(3)$ and $B$ be arbitrary, then $\Pi^{\mathfrak{g},B}$ is non-degenerate (see Fig. 8).

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + y^2 + z^2, \quad F_2 = ax + by + cz$

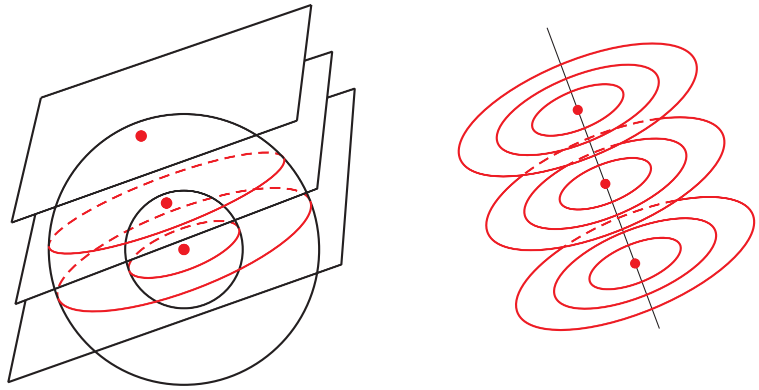

Figure 8: $so(3)$-case

**Example 10** (Semisimple case: $sl(2,\mathbb{R})$)**.** Consider the $sl(2,\mathbb{R})$–bracket $A$ and constant bracket $B$ defined by an element $\xi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2,\mathbb{R}) \simeq sl(2,\mathbb{R})^*$:

$$A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & 2a \\ b & -2a & 0 \end{pmatrix}$$

The Casimir functions of these brackets are respectively:

$$F_1 = x^2 + yz, \quad F_2 = 2ax + by + cz.$$

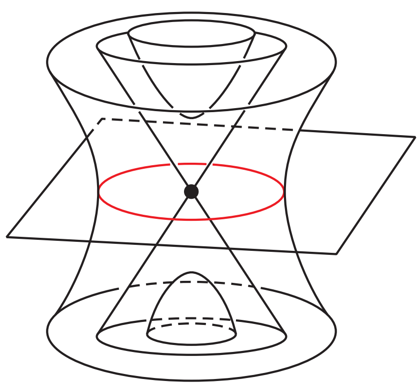

Figure 9: $sl(2)$ case

Fig. 9 shows that the singularity is non-degenerate and of elliptic type. However, if we take another $B$, the situation may change (see Fig. 10). Why are there 3 different cases? How to distinguish them? The point is that in $sl(2)$ there are non-trivial elements $\xi$ of three types:

- elliptic (eigenvalues are pure imaginary $i\lambda, -i\lambda$);

- hyperbolic (eigenvalues are real $\lambda, -\lambda$);
- nilpotent (both eigenvalues are zero).

We can distinguish them by using the Killing form:

- elliptic: $(\xi, \xi) < 0$;
- hyperbolic: $(\xi, \xi) > 0$;
- nilpotent: $(\xi, \xi) = 0$.

Equivalently, one may use the sign of $\det \xi$ in the standard representation.

This example leads us to the following conclusion:

$$\text{Non-degeneracy} \quad \Leftrightarrow \quad \xi \text{ is semisimple.}$$

**Example 11** (Non-semisimple case).

Consider the Lie algebra $e(2) = so(2) +_\phi \mathbb{R}^2 = \left\{ \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\}$

The corresponding Lie-Poisson bracket: $A = \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & 0 \\ \xi_2 & 0 & 0 \end{pmatrix}$

where $\xi_1, \xi_2, \xi_3$ are the dual coordinates to $x, y, z$.

Constant bracket: $B = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}$

Take the corresponding linear pencil $\Pi^{e(2),B}$.

Casimir functions on $e(2)^*$: $F_A = \xi_2^2 + \xi_3^2$, $F_B = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$.

These functions give a non-degenerate singularity iff $b_1 \neq 0$.

Algebraic reformulation:

$$\text{Non-degeneracy} \quad \Leftrightarrow \quad \operatorname{Ker} B \text{ is a Cartan subalgebra of } e(2).$$

**Example 12** (Semisimple Lie algebras of higher dimension).

Let $\mathfrak{g}$ be a semisimple Lie algebra different from $so(3)$ and $sl(2)$, for instance, $so(n)$ or $sl(n)$. As we know, the "argument shift" pencil gives a polynomial integrable system on this Lie algebra. Is the origin $0 \in \mathfrak{g} = \mathfrak{g}^*$ a non-degenerate critical point?

For the sake of simplicity, let $\mathfrak{g} = sl(4)$. The family of polynomial shifts in this case is "produced" from the polynomial invariants of the (co)adjoint representation, which are the functions of the form:

$$\operatorname{tr} X^2, \quad \operatorname{tr} X^3, \quad \operatorname{tr} X^4, \quad \text{where } X \in sl(4).$$

If we replace $X$ by $X + \lambda A$ and expand in powers of $\lambda$, we obtain 9 homogeneous commuting polynomials whose degrees are

$$4, 3, 3, 2, 2, 2, 1, 1, 1.$$

These are generators of the algebra $\mathcal{F}_A$ of polynomial shifts. In other words, each polynomial Casimir of degree $m$ generates $m$ commuting polynomials of degree $1, 2, \dots m-1, m$.

To verify the non-degeneracy condition at the origin we need to restrict these functions to the symplectic leaf of the constant bracket $\{\ ,\ \}_A$ (which is defined by three linear equations $h_1 = 0, h_2 = 0, h_3 = 0$, where $h_1, h_2, h_3$ are exactly the three linear polynomials from the set of shifts). Obviously, the origin is a critical point for all of them and, therefore, is indeed a common equilibrium. Is this singularity non-degenerate?

The answer is NO. The reason is very simple. A necessary (but not sufficient!) condition for non-degeneracy is that the Hessians of the commuting functions must be linearly independent. However, in our case the set of commuting functions contains two cubic polynomials and one of degree 4. Their Hessians vanish at the origin.

Thus, this example leads us to the conjecture that the Casimir functions of the Lie algebras we are interested in (i.e. with non-degenerate singularities) must be at most quadratic. This conjecture is indeed true.

## Classification of non-degenerate linear pencils

The following theorem describes all "good" Lie algebras $\mathfrak{g}$ (equivalently, Lie-Poisson brackets $A$) which may "produce" non-degenerate linear pencils and then states necessary and sufficient condition for a constant bracket $B$ on $\mathfrak{g}^*$ to give indeed a non-degenerate pencil $\Pi = \Pi^{\mathfrak{g},B}$. Such Lie algebras and linear pencils are called *non-degenerate*.

Classification in the complex case:

**Theorem 13** (A. Izosimov)**.** *A linear pencil $\Pi = \Pi^{\mathfrak{g},B}$ is non-degenerate (in the complex case) if and only if the Lie algebra $\mathfrak{g}$ is isomorphic to*

$$\left(\bigoplus so(3,\mathbb{C})\right) \oplus \left(\left(\bigoplus \mathfrak{D}\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{C}\right)$$

*where $\mathfrak{D}$ is the diamond Lie algebra, $\mathfrak{h}_0$ is a commutative ideal which belongs to the centre of $(\bigoplus \mathfrak{D})$, and* $\operatorname{Ker} B$ *is a Cartan subalgebra of* $\mathfrak{g}$.

This theorem basically states that all non-degenerate complex Lie algebras can be obtained from very "simple blocks", namely $so(3)$, $\mathfrak{D}$ and Abelian Lie algebras[5], by taking direct sums and quotienting w.r.t. a sub-centre.

[5]Recall that the complex Lie algebras $so(3,\mathbb{C})$ and $sl(2,\mathbb{C})$ are isomorphic, so the difference between them will be essential only in the real case.

What is the diamond Lie algebra $\mathfrak{D}$?

$\mathfrak{D}$ is a four dimensional Lie algebra generated by $e, f, t, h$ with the following relations

$$[t, e] = f, \quad [t, f] = -e, \quad [e, f] = h \quad \text{and} \quad [h, \mathfrak{D}] = 0. \tag{2}$$

In other words, $\mathfrak{D}$ (as a complex Lie algebra) is the non-trivial central extension of $e(2, \mathbb{C})$.
Matrix representation:

$$\alpha e + \beta f + \theta t + \gamma h \quad \mapsto \quad \begin{pmatrix} 0 & \alpha & \beta & 2\gamma \\ 0 & 0 & -\theta & \beta \\ 0 & \theta & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Casimir functions: $\qquad F_1 = f^2 + e^2 + 2th, \quad F_2 = h.$

Before stating the answer in the real case, we make a couple of remarks. First of all, it is the real case that is important in applications. From the dynamical viewpoint, the real case is much richer too. In integrable systems, we can observe elliptic, hyperbolic and focus singularities in different combinations and the dynamics of a system will essentially depend on the type of the singularity.

If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ is its complexification, then $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ are both degenerate or non-degenerate simultaneously. This easily follows from the simple observation that the family of shifts for $\mathfrak{g}^{\mathbb{C}}$ can be obtained from that of $\mathfrak{g}$ by replacing real variables $x_i$ by complex variables $z_i$. Since the non-degeneracy condition is in essence algebraic, this "operation" does not change anything. In other words, all real forms of a non-degenerate complex Lie algebra are non-degenerate too.

Finally, $k$ complex commuting functions on a complex symplectic manifold of dimension $n$ can be considered as $2k$ commuting functions on a real symplectic manifold of dimension $2n$. It is easy to see that the "non-degeneracy condition" is preserved under this passage from complex to real. Moreover, the real singularity will be of focus type (elliptic and hyperbolic components cannot appear in this way). This observation immediately implies that every non-degenerate $n$ dimensional complex Lie algebra $\mathfrak{g}$ treated as a $2n$-dimensional real Lie algebra will be non-degenerate in the real sense too.

Thus, in the real case, the elementary blocks from which non-degenerate Lie algebras can be built include the real forms of $so(3, \mathbb{C})$ and $\mathfrak{D}$, as well as the algebras $so(3, \mathbb{C})$ and $\mathfrak{D}$ themselves treated as real Lie algebras (of dimension 6 and 8 respectively).

The orthogonal Lie algebra $so(3, \mathbb{C})$ has 2 different real forms too:

- $so(3, \mathbb{R})$ and
- $sl(3, \mathbb{R})$

The complex diamond Lie algebra $\mathfrak{D}$ has 2 different real forms:

- $\mathfrak{g}_{ell}$ defined by (2) and
- $\mathfrak{g}_{hyp}$ defined by $[t,e]=e$, $[t,f]=-f$, and $[e,f]=h$.

The above remarks clarify the nature of the classification theorem below, but should not be considered as its formal justification.

**Theorem 14** (A. Izosimov)**.** *A real Lie algebra* $\mathfrak{g}$ *is non-degenerate iff*

$$\mathfrak{g} \simeq \Big(\bigoplus so(3,\mathbb{R})\Big) \oplus \Big(\bigoplus sl(2,\mathbb{R})\Big) \oplus \Big(\bigoplus so(3,\mathbb{C})\Big) \oplus \\ \Big(\Big(\Big(\bigoplus \mathfrak{g}_{ell}\Big) \oplus \Big(\bigoplus \mathfrak{g}_{hyper}\Big) \oplus \Big(\bigoplus \mathfrak{g}_{foc}\Big)\Big)/\mathfrak{h}_0\Big) \oplus \Big(\bigoplus \mathbb{R}\Big)$$

*where*

- $\mathfrak{g}_{ell}$ *and* $\mathfrak{g}_{hyp}$ *are the non-trivial central extensions of* $e(2)$ *and* $e(1,1)$ *(equivalently, they are real forms of* $\mathfrak{D}$*),*
- $\mathfrak{g}_{foc} = \mathfrak{D}$ *treated as real Lie algebra,*
- $\mathfrak{h}_0$ *is a commutative ideal which belongs to the centre.*

*A linear pencil* $\Pi^{\mathfrak{g},B}$ *is non-degenerate if* $\mathfrak{g}$ *is non-degenerate and* $\operatorname{Ker} B$ *is a Cartan subalgebra of* $\mathfrak{g}$*.*

The type of the singularity is naturally defined by the "number" of elliptic, hyperbolic and focus components in the above decomposition. The only exception is the Lie algebra $sl(2,\mathbb{R})$ for which, as we have seen in Example 10, two cases are possible, hyperbolic and elliptic, depending on the sign of the Killing form.

## General non-degeneracy criterion

Finally we explain how to verify the non-degeneracy of singularities for bi-Hamiltonian systems in the general case.

Let $\Pi = \{A+\lambda B\}$ be an arbitrary pencil of compatible Poisson brackets (of Kronecker type). We consider the commutative family of functions $\mathcal{F}_\Pi$ and a singular point $x \in \mathsf{S}_\Pi$. Our goal is to verify the non-degeneracy condition for $x$. Recall that we assume that almost all Poisson structures $A_\lambda \in \Pi$ have maximal rank at $x$.

We know already (see Lecture 2) that $x$ is singular for the family of commuting functions $\mathcal{F}_\Pi$, if and only if there is one or more values of the parameter $\lambda$ such that the rank of corresponding Poisson structure $A_\lambda = A + \lambda B$ drops at the point $x$. In algebraic terms, this means that the pencil $\Pi(x) = \{A(x) + \lambda B(x)\}$ of skew symmetric forms possesses a non-trivial spectrum $\Lambda(x) = \{\lambda_1, \dots, \lambda_k\}$ (see Jordan-Kronecker decomposition theorem, Lecture 2). On the other hand, from the geometric viewpoint we can reformulate this condition by saying that $x$ belongs to the singular sets $\mathsf{S}_{\lambda_1}, \dots, \mathsf{S}_{\lambda_k}$, where

$$\mathsf{S}_{\lambda_i} = \{y \in M \mid \operatorname{rank} A_{\lambda_i}(x) < \operatorname{rank} \Pi\}.$$

For each $\lambda_i \in \Lambda$, we can consider the $\lambda_i$-linearisation of the pencil $\Pi$. It turns out that these $\lambda_i$-linearisations contain enough information to verify the non-degeneracy condition for $x$.

**Theorem 15** (A. Izosimov)**.** *Let $\Pi = \{A + \lambda B\}$ be a pencil of compatible Poisson brackets, $\mathcal{F}_\Pi$ be the associated commutative family of functions and $x \in M$ singular point for $\mathcal{F}_\Pi$. This point is non-degenerate if and only if for every $\lambda_i \in \Lambda(x)$,*

1. *the $\lambda_i$-linearisation of the pencil $\Pi$ at $x$ is non-degenerate,*
2. *the corank of the $\lambda_i$-linearisation equals* $\operatorname{corank} \Pi$.

Comments for Theorem 15.

- The second condition can be interpreted in both algebraic and geometric terms. Algebraically, this means that the pencil $\Pi(x) = \{A(x) + \lambda B(x)\}$ is diagonalisable in the sense that the Jordan-Kronecker decomposition for $\Pi(x)$ contains no non-trivial Jordan blocks, i.e., all Jordan blocks are $2 \times 2$ and of the form $A_i + \lambda B_i = \begin{pmatrix} 0 & \lambda_i - \lambda \\ \lambda - \lambda_i & 0 \end{pmatrix}$ (there could be several blocks with the same $\lambda_i$, in other words, elements of the spectrum may have non-trivial multiplicities). Equivalently, one can say that the recursion operator, which can be naturally defined for the Jordan part of this decomposition, is diagonalisable.

  From the geometric viewpoint, condition 2 is equivalent to a very natural property of the (transversal) linearisation of $A_{\lambda_i}$ at the point $x$. It is easy to see the rank of the linearised Poisson structure $A_{\text{linear}}$ can, in general, be smaller than that of $A_{\text{transv}}$. The second condition simply means that after the linearisation, the rank of $A_{\lambda_i}$ does not drop but remains the same. In other words, the rank is preserved under linearisation.

- The type of the singularity $x \in \mathsf{S}_\Pi$ is just the "sum" of types of the $\lambda_i$-linearisations. Recall that each $\lambda_i$-linearisation is represented as a finite-dimensional Lie algebra $\mathfrak{g}$ with a cocycle $B$. The type of the singularity for $\Pi^{\mathfrak{g},B}$ is basically defined by the types of components in the decomposition of $\mathfrak{g}$ into elementary blocks (see Theorem 14). The only exception is the $sl(2)$-block that requires some additional analysis involving $B$.

- The spectrum $\Lambda(x)$ may contain pairs of complex conjugate numbers $\alpha_j \pm i\beta_j$. The $\lambda_j$ and $\bar\lambda_j$-linearisations will be represented by complex Lie algebras and they will contribute as focus components.

- As we know, stable singularities are of purely elliptic type. Thus, stability of bi-Hamiltonian systems is, in essence, determined by the algebraic type of the $\lambda_i$-linearisations (see Theorem 14). Again the only exceptions are $sl(2)$-components which may "produce" both hyperbolic and elliptic singularities.

- This theorem provides a very useful tool which essentially simplifies the analysis of singularities of bi-Hamiltonian systems. Indeed, following the standard scheme we need to

  1. choose some basis $f_1, \dots, f_s$ in the family of first integrals,
  2. compute the Jacobi matrix $J = \left(\frac{\partial f_i}{\partial x_k}\right)$ in appropriate local coordinates and find those points where the rank of $J$ drops,
  3. and, finally, analyse the Hessians $J = \left(\frac{\partial^2 f_i}{\partial x_k \partial x_j}\right)$ at such points.

  Already for two or three degrees of freedom systems, this straightforward approach requires very non-trivial (but yet reasonable) analytic computations. For many degrees of freedom, technical difficulties becomes unsurmountable and we need to use some additional hidden structure. That is exactly the main idea of this approach. All the information we need is hidden in the algebraic properties of the underlying bi-Hamiltonian structure and Theorem 15 explains how to uncover it. Instead of analysing the differentials of integrals, we only need to study the singular sets of the Poisson brackets involved. Since these brackets usually have an algebraic nature, the questions we are interested in can often be treated by purely algebraic means. The next lecture is devoted to examples showing how to use the above techniques in practice.

# References


- Poisson structure: splitting and linearisation

  - Alan Weinstein. The local structure of Poisson manifolds. J. Differential Geom., Volume 18, Number 3, (1983), 523–557.
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- Non-degeneracy:

  - A. Bolsinov, A. Izosimov, Singularities of bi-Hamiltonian systems, arXiv:1203.3419v2.

# Lecture 4:
# How does it work? Examples and applications

**Four examples.**

- Rubanovskii case in rigid body dynamics
- Mischenko-Fomenko systems on semisimple Lie algebras
- Euler-Manakov top on $so(n)$
- Periodic Toda lattice

Some of the results presented below are well known to experts in the field and were obtained quite a while ago. We use them as an illustration of the bi-Hamiltonian approach that essentially simplifies original proofs.

We also would like to point out one important feature of the method we are using. Our main goal is studying qualitative properties of a dynamical system (e.g., stability of solutions) as well as the properties of the singular Lagrangian fibration defined by the commuting integrals of the system. It might look strange but for our analysis we need neither the equation of motion nor explicit formulas for the integrals. The only important information is the Poisson pencil associated with the given system.

## Rubanovskii case

This integrable system is a generalisation of the famous Steklov–Lyapunov case of the Kirchhoff equations discovered by V.N. Rubanovskii.

The most convenient and elegant way to describe this system is the Lax pair found by Yu. Fedorov:

$$\frac{dL(\lambda)}{dt} = [L(\lambda), A(\lambda)], \quad L(\lambda), A(\lambda) \in so(3), \quad \lambda \in \mathbb{C},$$

where

$$L_{\alpha\beta}(\lambda) = \varepsilon_{\alpha\beta\gamma} \left( \sqrt{\lambda - b_\gamma}\, (z_\gamma + \lambda p_\gamma) + g_\gamma / \sqrt{\lambda - b_\gamma} \right),$$
$$A(\lambda)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} \frac{1}{\lambda} \sqrt{(\lambda - b_\alpha)(\lambda - b_\beta)} (b_\gamma z_\gamma - g_\gamma)$$

and $\varepsilon_{\alpha\beta\gamma}$ denotes the Levi-Civita symbol. In this representation $z$ and $p$ are dynamical variables, and $b$ and $g$ are geometric parameters.

The analysis of singularities, based on the bi-Hamiltonian approach, was carried out by I. Basak (PhD thesis, Universitat Politecnica de Catalunya).

The initial point for this analysis is a description of the corresponding pencil of compatible Poisson brackets.

**Proposition 5.** *The Rubanovskii system is Hamiltonian w.r.t. the pencil defined on $\mathbb{R}^6(z,p)$ by the following compatible Poisson brackets:*

$$P_0 = \begin{pmatrix} 0 & b_3z_3 - g_3 & -b_2z_2 + g_2 & 0 & 0 & 0 \\ -b_3z_3 + g_3 & 0 & b_1z_1 - g_1 & 0 & 0 & 0 \\ b_2z_2 - g_2 & -b_1z_1 + g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & 0 & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & 0 & p_2 & -p_1 & 0 \end{pmatrix}$$

*and*

$$P_\infty = \begin{pmatrix} 0 & z_3 - b_3p_3 & -z_2 + b_2p_2 & 0 & p_3 & -p_2 \\ -z_3 + b_3p_3 & 0 & z_1 - b_1p_1 & -p_3 & 0 & p_1 \\ z_2 - b_2p_2 & -z_1 + b_1p_1 & 0 & p_2 & -p_1 & 0 \\ 0 & p_3 & -p_2 & 0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & 0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

*where $z,p$ are coordinates in the phase space $\mathbb{R}^6$, $b$ and $g$ are geometric parameters.*

The algebraic structure of $P_0 - \lambda P_\infty$ becomes clear when we change variables:

$$\tilde{z}_i = z_i + \lambda p_i + \frac{g_i}{\lambda - b_i}, \qquad p_i\text{'s remain the same}$$

After this change of variables, the pencil reads:

$$P_0 - \lambda P_\infty = \begin{pmatrix} 0 & (b_3-\lambda)\tilde{z}_3 & -(b_2-\lambda)\tilde{z}_2 & & & \\ -(b_3-\lambda)\tilde{z}_3 & 0 & (b_1-\lambda)\tilde{z}_1 & & & \\ (b_2-\lambda)\tilde{z}_2 & -(b_1-\lambda)\tilde{z}_1 & 0 & & & \\ & & & 0 & p_3 & -p_2 \\ & & & -p_3 & 0 & p_1 \\ & & & p_2 & -p_1 & 0 \end{pmatrix}$$

Thus, $P_1 - \lambda P_0$ splits into the direct sum of two brackets, one of which is the standard $so(3)$-bracket and the other is isomorphic to either $so(3)$, or $sl(2)$ depending on the signs of $b_i - \lambda$, $i = 1,2,3$.

Question: What are the critical points for the integrals? The answer is universal: those points where the rank of $P_0 - \lambda P_\infty$ drops. So we obtain

**Theorem 16.** *A point $(z,p)$ is critical if and only off there is $\lambda \in \mathbb{C} \setminus \{b_1, b_2, b_3\}$ such that*

$$z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0, \qquad i = 1,2,3.$$

Common equilibrium points are easy to describe too.

**Theorem 17.** *A point $(z,p)$ is a common equilibrium if and only if*

$$\operatorname{rank}\begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} = 1.$$

The non-degeneracy condition for corank one singularities is easy to obtain by using the $\lambda$-linearisation techniques.

**Theorem 18.** *Let $\gamma$ be a critical closed trajectory passing through $(z,p)$ with parameter $\lambda$. Then $\gamma$ is non-degenerate if and only if*

$$C = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3) \sum_{i=1}^{3} \left( (\lambda - b_i) p_i - \frac{g_i}{\lambda - b_i} \right)^2 \frac{1}{\lambda - b_i} \neq 0$$

*Moreover, if $C > 0$ then $\gamma$ is stable, and if $C < 0$ then $\gamma$ is unstable.*

For corank zero singularities (i.e. common equilibria) the result is similar. Notice that all these results are obtained by straightforward application of general theorems discussed in Lectures 2 and 3 to the pencil $P_0 - \lambda P_\infty$.

## Mischenko-Fomenko systems on semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional (real) Lie algebra and $g^*$ its dual space endowed with two Lie–Poisson brackets:

$$\{f, g\}(x) = x([df(x), dg(x)]) \quad \text{and} \quad \{f, g\}_a(x) = a([df(x), dg(x)]),$$

where $f, g : \mathfrak{g}^* \to \mathbb{R}$ are arbitrary smooth functions, $x, a \in \mathfrak{g}^*$ and $a$ is fixed and regular. This is a typical example of a linear pencil (see Lecture 3).

The pencil $\{\ ,\ \} + \lambda\{\ ,\ \}_a$ leads to the family of commuting Casimirs

$$\mathcal{F}_a = \{ f(x + \lambda a) \mid f \text{ is a Casimir of } \{\ ,\ \}, \lambda \in \mathbb{R} \}.$$

Below we discuss the properties of $\mathcal{F}_a$ for semisimple (and even compact) Lie algebras. In this case $\mathfrak{g} \simeq \mathfrak{g}^*$ and the family $\mathcal{F}_a$ possesses a natural basis consisting of $s = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ homogeneous polynomials $f_1, \ldots, f_s$. In other words, $\mathcal{F}_a$ is freely generated by them.

MF systems on $\mathfrak{g}$ can be understood as Hamiltonian systems with quadratic Hamiltonians $H \in \mathcal{F}_a$. It can be shown that they are automatically bi-Hamiltonian w.r.t. the pencil $\{\ ,\ \} + \lambda\{\ ,\ \}_a$. We want to study the properties of the corresponding momentum map $\Phi_a : \mathfrak{g} \to \mathbb{R}^s$, $\Phi_a(x) = (f_1(x), \ldots, f_s(x))$, by applying the bi-Hamiltonian approach presented in Lectures 2 and 3.

**Theorem 19** (Mischenko, Fomenko, 1976)**.** *If $\mathfrak{g}$ is semisimple and $a \in \mathfrak{g}^*$ is regular, then the collection of commuting polynomials $\mathcal{F}_a$ is complete on $\mathfrak{g} \simeq \mathfrak{g}^*$. In other words, $f_1, \ldots, f_s$ are functionally independent on $\mathfrak{g}$.*

This theorem is a particular case of the following general result.

**Theorem 20.** *In the case of an arbitrary finite dimensional* $\mathfrak{g}$*, the family* $\mathcal{F}_a$ *is complete, i.e. contains* $s = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$ *functionally independent functions, if and only if* $\operatorname{codim} \mathsf{Sing} \geq 2$*, where* $\mathsf{Sing} \subset \left(\mathfrak{g}^{\mathbb{C}}\right)^*$ *is the set of singular elements.*

The proof is obvious. We just need to apply the codimension two principle and use, in the semisimple case, the well-known algebraic fact that $\operatorname{codim} \mathsf{Sing} = 3$ for all semisimple Lie algebras.

**Theorem 21.** *An element* $x \in \mathfrak{g}$ *is a critical point of the momentum map* $\Phi_a$ *if and only if there exists* $\lambda \in \mathbb{C}$ *such that* $x + \lambda a$ *is a singular element in* $\mathfrak{g}^{\mathbb{C}}$*. In other words, the set of critical points* $\mathsf{S}_a$ *of* $\Phi_a$ *is (the real part of) the cylinder over the set of singular elements* $\mathsf{Sing}$ *with the generating line parallel to* $a$*, that is:* $\mathsf{S}_a = (\mathsf{Sing} + \mathbb{C} \cdot a) \cap \mathfrak{g}$.

Recall that $x \in \mathfrak{g}$ is said to be a common equilibrium point for $\mathcal{F}_a$, if for all $f \in \mathcal{F}_a$ we have $X_f(x) = [df(x), x] = 0$.

**Theorem 22.** *A point* $x \in \mathfrak{g}$ *is a common equilibrium point for* $\mathcal{F}_a$ *if and only if* $x \in \mathfrak{h}_a$*, where* $\mathfrak{h}_a$ *is the Cartan subalgebra generated by* $a \in \mathfrak{g}$*. The number of equilibrium points on each regular orbit is the order of the Weil group.*

Let $x \in \mathfrak{h}_a$ be a common equilibrium point. Is $x$ non-degenerate?

**Theorem 23** (Sufficient condition)**.** *Let* $\alpha_1, \dots, \alpha_s$ *be the positive roots associated with the complexification* $\mathfrak{h}_a^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$*,* $s = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g})$*. Consider the collection of numbers*

$$\lambda_i = \frac{\alpha_i(x)}{\alpha_i(a)}$$

*If all these numbers are distinct, then* $x \in \mathfrak{g}$ *is a non-degenerate equilibrium point. Moreover, if* $\mathfrak{g}$ *is compact, then* $x$ *is of purely elliptic type and, therefore, is stable.*

*Proof.* The proof easily follows from the general non-degeneracy criterion. The only thing we need to do is to analyse $\lambda_i$-linearisations of the pencil at the point $x \in \mathfrak{h}_a$ for each $\lambda_i$ from the spectrum $\Lambda(x)$. Recall that $\lambda \in \Lambda(x)$ if and only if $x + \lambda a$ is a singular point. Since $x, a$ both belong to the Cartan subalgebra $\mathfrak{h}_a$, the element $x+\lambda a$ becomes singular if for some root $\alpha_i$ we have $\alpha_i(x + \lambda a) = 0$, i.e. $\lambda = -\lambda_i = -\frac{\alpha_i(x)}{\alpha_i(a)}$. The $-\lambda_i$-linearisation in this case is easy to describe. The Lie algebra $\mathfrak{g}_{-\lambda_i}$ is the centraliser of $x - \lambda_i a$:

$$\mathfrak{g}_{-\lambda_i} = \{\xi \in \mathfrak{g} \mid [\xi, x - \lambda_i a] = 0\}$$

and the 2-form on this centraliser is defined by the same element $a \in \mathfrak{g}_{-\lambda_i}$.

If all $\lambda_i$ are distinct, then $\alpha_i$ is the only root that vanishes at $x + \lambda_i a$ and, therefore, $\mathfrak{g}_{-\lambda_i} = \operatorname{span}\{\mathfrak{h}_a, e_{\alpha_i}, e_{-\alpha_i}\}$, i.e. is isomorphic to either $so(3) + \mathbb{R}^{r-1}$, or $sl(2) + \mathbb{R}^{r-1}$, or $so(3, \mathbb{C}) + \mathbb{C}^{r-1}$ (if $\lambda_i \in \mathbb{C}$) and the kernel of the 2-form is still the same Cartan subalgebra $\mathfrak{h}_a$. Thus, this linearisation is obviously non-degenerate for each $\lambda_i$ and, hence, $x$ is a non-degenerate singular point. □

What about corank 1 singularities?
Let $x \in \mathfrak{g}$ be a critical point of corank 1 of the momentum map $\Phi_a$, then

- there is a unique value of the parameter $\lambda \in \mathbb{R}$ such that $x + \lambda a$ is a singular element of $\mathfrak{g}$;
- the centraliser $\mathfrak{g}_\lambda$ of $x + \lambda a$ has dimension $\operatorname{ind} \mathfrak{g} + 2$;

If we assume that $x + \lambda a$ is a semisimple element (this situation is generic), then the centraliser $\mathfrak{g}_\lambda$ of $x + \lambda a$ is isomorphic to the direct sum $\mathfrak{u} \oplus \mathbb{R}^{r-1}$, where $\mathfrak{u}$ is either $so(3)$ or $sl(2)$. According to the general scheme, we now should take the restriction of the $a$-bracket onto $\mathfrak{g}_\lambda$ and verify if the kernel of this restriction is a Cartan subalgebra. As we already discussed, this restriction is defined by the natural projection of $a$ onto $\mathfrak{g}_\lambda^*$. Since the centre $\mathbb{R}^{r-1}$ of $\mathfrak{g}_\lambda$ does not play any role in this construction, we come to the following conclusion.

**Theorem 24.** *Let $x \in \mathfrak{g}$ be a critical point of corank 1 of the momentum map $\Phi_a$ and $\lambda \in \mathbb{R}$ the unique value of the parameter such that $x+\lambda a$ is a singular element of $\mathfrak{g}$. Assume that $x + \lambda a$ is semisimple and $\mathfrak{u}$ is the semisimple part of the centralizer of $x + \lambda a$. Consider the natural orthogonal projection $b = \mathrm{pr}_{\mathfrak{u}}\, a$ of $a$ onto $\mathfrak{u}$. Then $x$ is non-degenerate if and only if $b \in \mathfrak{u}$ is semisimple and non-zero.*

*Moreover, if $(b, b) > 0$, then the singularity is hyperbolic, and if $(b, b) < 0$, then the singularity is elliptic, where $(\ ,\ )$ is the Killing form on $\mathfrak{u}$.*

In particular, in the case of a compact Lie algebra $\mathfrak{g}$, all corank 1 singularities are non-degenerate and of elliptic type. In this case, there are no hyperbolic singularities. It follows from this that the set of regular values of $\Phi_a$ in $\mathbb{R}^s$ is connected and each non-trivial regular level $\Phi_a^{-1}(y)$, $y \in \mathbb{R}^s$, consists of one Liouville torus.

**Remark 4.** The assumption that $x + \lambda a$ is semisimple is not necessary for the non-degeneracy of $x$. For some non-semisimple elements $x + \lambda a$, the centraliser $\mathfrak{g}_{x+\lambda a}$ may have the form $\mathfrak{D} \oplus \mathsf{centre}$ which is still a non-degenerate Lie algebra.

## Euler-Manakov tops on $so(n)$

The Euler-Manakov top on $so(n)$ is an $n$-dimensional generalisation of the classical Euler equations in rigid body dynamics. The E-M equations are

$$\frac{d}{dt}X = [R(X), X], \quad X \in so(n),$$

where $R : so(n) \to so(n)$ is a linear operator which, in terms of matrix coefficients, takes the form $R(X)_{ij} = \dfrac{b_i - b_j}{a_i - a_j} X_{ij}$.

As was shown by S.V.Manakov, this system admits a family of commuting integrals of the form $\mathrm{tr}(X+\lambda A)^k$, where $A = \mathrm{diag}(a_1, \dots, a_n)$ is the diagonal matrix with diagonal elements $a_i$. The fact that this family is complete (i.e. sufficient for Liouville integrability) was proved by A.S.Mischenko and A.T.Fomenko. Our aim is to study the singularities of this integrable system.

First of all, we need to describe the bi-Hamiltonian structure for the E-M top. Along with the standard commutator $[X, Y] = XY - YX$ on the space of skew-symmetric matrices, we introduce a new operation

$$[X, Y]_A = XAY - YAX$$

where $A$ is a symmetric matrix. It is easy to see that these two brackets are compatible in the sense that their linear combination $[\ ,\ ]_A + \lambda[\ ,\ ] = [\ ,\ ]_{A+\lambda E}$ satisfies the Jacobi identity, so that on the dual space $so(n)^*$ we obtain a pencil of compatible linear Poisson brackets.

It turns out that the E-M top is Hamiltonian w.r.t. this pencil. The Casimirs of the bracket $\{\ ,\ \}_{A+\lambda E}$ are of the form

$$\mathrm{Tr}\left(X(A+\lambda E)^{-1}\right)^k$$

and it is not difficult to verify that these integrals are equivalent to those found by Manakov:

$$\mathcal{F}_A = \left\{\mathrm{Tr}(X+\lambda A)^k\right\}.$$

This family admits a basis that consists of exactly $s = \frac{1}{2}\big(\dim so(n) + \mathrm{ind}\, so(n)\big)$ commuting polynomials.

**Theorem 25** (A.S.Mischenko, A.T.Fomenko)**.** *If the eigenvalues of $A$ are all distinct, then the family of Manakov's integrals $\mathcal{F}_A$ is complete on $so(n)$.*

*Proof (bi-Hamiltonian version).* Almost all brackets in the family are semisimple, so the singular set of each of them has codimension 3. The corank of those brackets which are not semisimple coincides with the corank of the pencil. Thus, the statement immediately follows from the codimension two principle (see Lecture 2).

**Theorem 26.** *$X \in so(n)$ is a critical point of $\mathcal{F}_A$ if and only if there exists $\lambda \in \overline{\mathbb{C}}$ such that $X$ is singular for the bracket $\{\ ,\ \}_{A+\lambda E}$. Equivalently,*

$$\mathsf{S}_A = \left(\bigcup_{i=1}^{n} \mathsf{S}_{-a_i}\right) \cup \left(\bigcup_{\lambda\in\bar{\mathbb{C}},\ \lambda\neq -a_i} (A+\lambda E)^{1/2}\,\mathsf{Sing}\,(A+\lambda E)^{1/2} \cap so(n,\mathbb{R})\right),$$

*where $\mathsf{Sing} \subset so(n,\mathbb{C})$ is the set of singular points and $\mathsf{S}_{-a_i}$ is the singular set of the bracket $\{\ ,\ \}_{A-a_iE}$.*

*Proof.* The first part is just a reformulation of Theorem 10 for this particular pencil. The second part follows from the observation that for $\lambda \neq -a_i$, the bracket $[\ ,\ ]_{A+\lambda E}$ is isomorphic to the standard $so(n)$ bracket and $X \mapsto (A+\lambda E)^{1/2} X (A+\lambda E)^{1/2}$ is the corresponding (generally speaking, complex!) isomorphism for the dual spaces. □

Notice that for $\lambda = -a_i$ the bracket $\{\ ,\ \}_{A-a_iE}$ becomes isomorphic to the $e(n-1)$ bracket. This passage from $so(n)$ to $e(n-1)$ as $\lambda \to -a_i$ is known as the *contraction* between these two Lie algebras and one might expect that their singular sets are closely related. By continuity, $\mathsf{S}_\lambda$ should "converge" to some part of $\mathsf{S}_{-a_i}$ as $\lambda \to -a_i$ and therefore, $(A-a_iE)^{1/2}\,\mathsf{Sing}\,(A-a_iE)^{1/2} \subset \mathsf{S}_{-a_i}$. It would be interesting to understand the relationship between these two sets in detail. In low dimensions, this can be done by a straightforward computation.

In dimension 3, for example, $\mathsf{S}_\lambda = \{0\}$ for $\lambda \neq a_i$, whereas $\mathsf{S}_{-a_i}$ is a line.

However in dimension 4, the set $\mathsf{S}_{-a_i}$ is the limit of $\mathsf{S}_\lambda$ as $\lambda \to -a_i$, in particular, $(A-a_iE)^{1/2}\,\mathsf{Sing}\,(A-a_iE)^{1/2} = \mathsf{S}_{-a_i}$. Indeed, $so(4) = so(3) \oplus so(3)$ and the set of singular points $\mathsf{Sing}$ in the complex Lie algebra $so(4,\mathbb{C})$ is the union of two 3-dim subspaces

$$V_1 = \begin{pmatrix} 0 & z_3 & -z_2 & z_1 \\ -z_3 & 0 & z_1 & z_2 \\ z_2 & -z_1 & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 & -z_3 & z_2 & z_1 \\ z_3 & 0 & -z_1 & z_2 \\ -z_2 & z_1 & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0 \end{pmatrix}$$

Under the transformation $X \mapsto (A+\lambda E)^{1/2} X (A+\lambda E)^{1/2}$, these two subspaces "move" and for $\lambda = -a_i$ coincide (at this very moment the transformation ceases to be invertible). This one single subspace is exactly the singular set $\mathsf{S}_{-a_i}$.

As a result, the description of the set of critical points for $\mathcal{F}_A$ becomes simpler:

$$\mathsf{S}_A = \bigcup_{i=1,2,\ \lambda\in\bar{\mathbb{C}}} V_i^\lambda,$$

where $V_i^\lambda = (A+\lambda E)^{1/2}\, V_i\, (A+\lambda E)^{1/2}$. This gives a natural parametrization for $\mathsf{S}_A$ by means of 4 parameters $z_1, z_2, z_3, \lambda$.

The next theorem describes the (common) equilibria for the E-M system.

**Theorem 27.** *The set of common equilibrium points of $\mathcal{F}_A$ (with $A$ diagonal) is the union of those Cartan subalgebras $\mathfrak{h} \subset so(n)$ which are common Cartan subalgebras for all commutators $[\ ,\ ]_{A+\lambda E}$. One of these Cartan subalgebras is standard:*

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} 0 & x_{12} & & & \\ -x_{12} & 0 & & & \\ & & 0 & x_{34} & \\ & & -x_{34} & 0 & \\ & & & & \ddots \end{pmatrix}, \quad x_{i,i+1} \in \mathbb{R} \right\}.$$

*All the others are obtained from $\mathfrak{h}_0$ by conjugation $\mathfrak{h}_0 \mapsto P\mathfrak{h}_0 P^{-1}$ where $P$ is a permutation matrix.*

This result was obtained by L.Fehér and I.Marshall. We give a bi-Hamiltonian version of the proof.

*Proof.* According to Theorem 11, common equilibria for $\mathcal{F}_A$ are those points where the kernels of all the brackets coincide. For the $so(n)$ bracket, this kernel at a regular point $X$ is the Cartan subalgebra containing $X$. Thus, we need to find common Cartan subalgebras for all the brackets $[\ ,\ ]_{A+\lambda E}$. An explicit description of such subalgebras is an easy algebraic exercise. □

We finally describe non-degenerate equilibria. Let $X$ be a $2\times 2$ block-diagonal skew-symmetric matrix (as above). For each pair of blocks we set $\begin{pmatrix} 0 & x_{i,i+1} \\ -x_{i,i+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, $\begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and consider the function

$$f(x) = \frac{(x-\lambda_1^2)(x-\lambda_2^2)}{\omega^2(\lambda_1+\lambda_2)^2}$$

formally assuming that $f(\infty) = \frac{1}{\omega^2(\lambda_1+\lambda^2)^2}$.

By drawing the graphs of all of these functions on the same plane $\mathbb{R}^2$, we obtain a collection of parabolas called the *parabolic diagram* $\mathcal{P}$. For simplicity we assume that $n$ is even. We say that this diagram is generic if any two parabolas intersect exactly at two points (including complex intersections and intersections at infinity)

**Theorem 28** (A. Izosimov)**.**

1. *The equilibrium point $X \in so(n)$ is non-degenerate if and only if the parabolic diagram $\mathcal{P}$ is generic, i.e.:*
    - *each intersection point in the upper half plane corresponds to an elliptic component;*
    - *each intersection point in the lower half plane corresponds to a hyperbolic component;*
    - *each complex intersection corresponds to a focus component.*
2. *If $\mathcal{P}$ is generic, all intersections are real and located in the upper half plane, then the equilibrium is stable.*
3. *If there is either a complex intersection or an intersection point in the lower half plane, then the equilibrium point is unstable.*

## Periodic Toda lattice

The phase space $M_T$ of the periodic Toda lattice with $n$ sites is $\mathbb{R}^n_+ \times \mathbb{R}^n$ endowed with Flaschka variables

$$a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+, \quad b = (b_1, \ldots, b_n) \in \mathbb{R}^n.$$

It is convenient to treat $a$ and $b$ as infinite $n$-periodic sequences

$$a \in \mathbb{R}^\infty_+,\ a_{i+n} = a_i, \quad b \in \mathbb{R}^\infty,\ b_{i+n} = b_i.$$

The equations of motion are

$$\begin{cases} \dot{a}_i = a_i(b_{i+1} - b_i), \\ \dot{b}_i = 2(a_i^2 - a_{i-1}^2). \end{cases}$$

It is well known that these equations are bi-Hamiltonian. The corresponding pencil $\Pi_T = \{P_0 + \lambda P_\infty\}$ is given by

$$\{a_i, b_i\}_0 = a_i b_i, \quad \{a_i, b_{i+1}\}_0 = -a_i b_{i+1}, \quad \{a_i, a_{i+1}\}_0 = -\frac{1}{2} a_i a_{i+1}, \quad \{b_i, b_{i+1}\}_0 = -2a_i^2,$$
$$\{a_i, b_i\}_\infty = a_i, \quad \{a_i, b_{i+1}\}_\infty = -a_i.$$

The corresponding Hamiltonians are

$$H_\lambda = \begin{cases} \sum_{i=1}^n b_i & \text{for } \lambda \neq \infty, \\ \sum_{i=1}^n a_i^2 + \frac{1}{2}\sum_{i=1}^n b_i^2 & \text{for } \lambda = \infty. \end{cases}$$

The integrals of the Toda lattice are the Casimir functions of the pencil $\Pi_T$. Taking all of these Casimirs, we obtain a complete family of polynomials $\mathcal{F}_T$ in bi-involution. Our goal is to study the singularities of $\mathcal{F}_T$.

According to the general scheme, in order to study the singularities of $\mathcal{F}_T$, we need to do the following.

1. For each point $x \in M_T$ determine the spectrum of the pencil at $x$. The point $x$ is singular if and only if the spectrum is non-empty.

2. If $x$ is singular, then for each $\lambda$ in the spectrum, check the following conditions:

   - $\dim \operatorname{Ker} \left(P_\infty(x)\,|_{\operatorname{Ker} P_\lambda(x)}\right) = \operatorname{corank} \Pi_T(x)$;
   - the linearized pencil $\mathrm{d}_\lambda \Pi_T(x)$ is non-degenerate.

   The point $x$ is non-degenerate if and only if these conditions are satisfied for each $\lambda$ in the spectrum.

3. If $x$ is non-degenerate, determine its type by adding up the types of $\mathrm{d}_\lambda \Pi(x)$.

First of all, notice that each Poisson structure $P_\lambda \in \Pi_T(x)$ is of corank two. Moreover, $P_\infty$ is of constant corank 2 (recall that we assume that $a_i > 0$).

Next, the map $(a, b) \mapsto (a, b - \lambda)$ transforms the bracket $P_\lambda$ into $P_0$. Thus, all these brackets are isomorphic. This observation reduces the study of singularities of the pencil to the singularities of $P_0$. To that end, consider the infinite Lax matrix

$$\mathcal{L}(a,b) = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a_{i-1} & b_i & a_i & & \\ & & a_i & b_{i+1} & a_{i+1} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

**Proposition 6.** *The singular set $\mathsf{S}_0$ of the Poisson structure $P_0$ consists of those infinite periodic sequences $a, b \in \mathbb{R}^\infty$ for which the equation $\mathcal{L}(a,b)\,\xi = 0$ has either two periodic or two anti-periodic solutions, i.e. zero is a multiplicity-two periodic or anti-periodic eigenvalue of $\mathcal{L}(a,b)$.*

*If $(a, b) \in \mathsf{S}_0$, then* $\operatorname{corank} P_0(a,b) = 4$ *and the transversal linearisation of $P_0$ at the point $(a,b) \in \mathsf{S}_0$ is isomorphic to the Lie algebra $sl(2,\mathbb{R}) \oplus \mathbb{R}$ (in particular, this Lie algebra is non-degenerate!).*

Thus, the structure of the singular set $\mathsf{S}_0$ is quite simple (as well as that of $\mathsf{S}_\lambda$ since $P_0$ and $P_\lambda$ are isomorphic).

The next step is to restrict the Poisson structure $P_\infty$ onto $\operatorname{Ker} P_0 \simeq (sl(2,\mathbb{R}) \oplus \mathbb{R})^*$. It can be shown by a straightforward computation that the kernel of this restriction is a Cartan subalgebra of $sl(2,\mathbb{R}) \oplus \mathbb{R}$ and moreover this subalgebra is of elliptic type. If we replace $P_0$ by $P_\lambda$, all the conclusions remain unchanged. The only difference is that the equation $\mathcal{L}(a,b)\,\xi = 0$ should be replaced by $\mathcal{L}(a,b)\,\xi = \lambda\xi$ so that $\lambda$ must be a multiplicity-two periodic or anti-periodic eigenvalue of $\mathcal{L}(a,b)$.

Thus, the structure of singularities of the family of commuting Casimirs $\mathcal{F}_T$ can be described as follows. If $\mathcal{L}(a,b)$ has no multiple periodic or anti-periodic eigenvalues, then the integrals of the Toda lattice are independent at the point $(a,b) \in M_T$. If $\mathcal{L}(a,b)$ has $k$ multiplicity-two periodic or anti-periodic eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then these numbers $\lambda_i$'s are exactly the elements of the spectrum $\Lambda$ of the pencil of compatible Poisson structures $P_\lambda$ at the point $(a,b) \in M_T$. Since each $\lambda_i$-linearisation is non-degenerate and elliptic, we come to the following

**Theorem 29.** *All singularities of the periodic Toda lattice are non-degenerate and of elliptic type. In particular, all of them are (orbitally) stable.*

By using different methods, this result was obtained by J.A. Foxman and J.M. Robbins.

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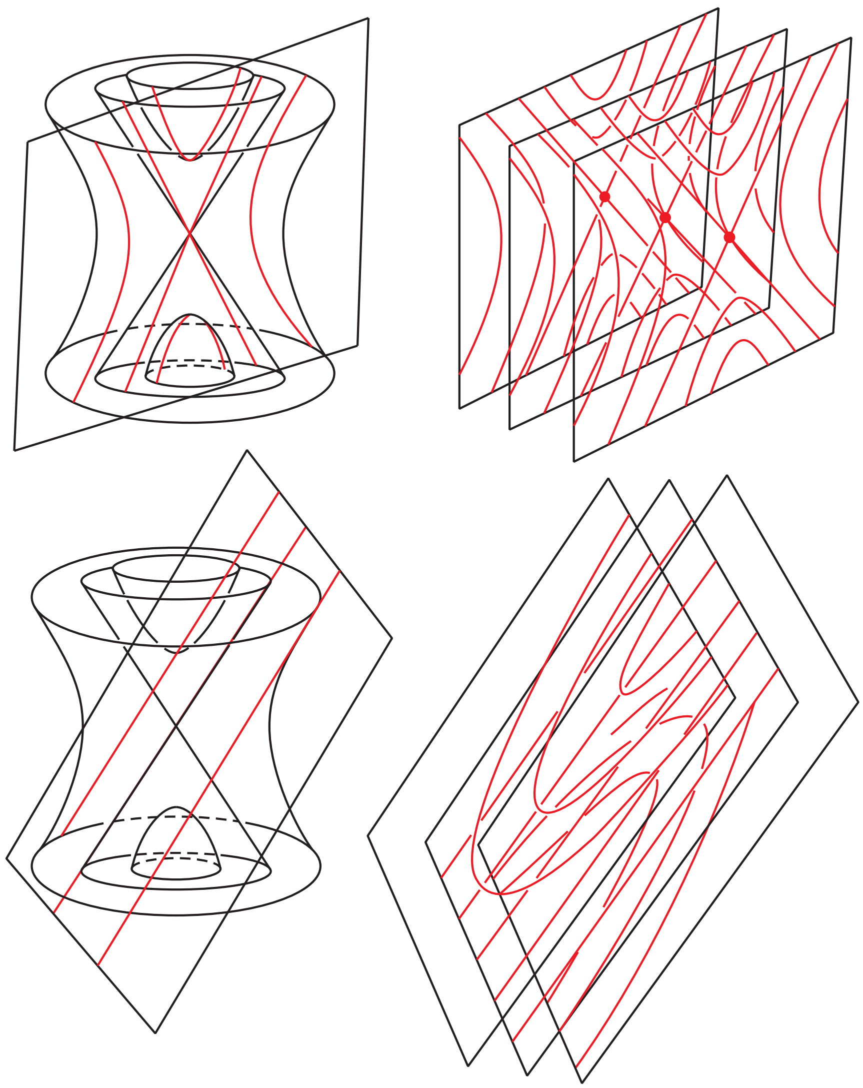

Figure 10: $sl(2)$ case: two other possibilities