Stability analysis and bi-Hamiltonian systems

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Some basic notions and notation

Symplectic manifold \((M, \omega)\)

Hamiltonian system \(\dot{x} = X_H(x) = \omega^{-1}(dH(x))\)

Integrability: there exist \(f_1, \ldots, f_n : M \to \mathbb{R}\) which:

- first integrals of \(X_H(x)\);
- commute;
- independent almost everywhere.

Singular Lagrangian fibration on \(M\) whose generic fibers are Liouville tori with quasi-periodic dynamics

Set of critical points \(S = \{x \in M \mid \text{rank}(df_1(x), \ldots, df_n(x)) < n\}\)

SINGULARITIES ARE IMPORTANT

General problem: Describe \(S\) and its properties. In particular, find all stable periodic orbits and equilibria.
Some basic notions and notation

Poisson manifold \((M, A)\), Poisson structure \(A = (A^{ij})\) and Poisson bracket
\[
\{f, g\}_A = A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.
\]
We set \(\text{rank } A = \max_{x \in M} \text{rank } A(x)\).
If \(\text{rank } A < \dim M\) then, as a rule, there exist Casimir functions \(f \in C^\infty(M)\) such that
\[
\{f, g\}_A = 0 \quad \text{for any } g \in C^\infty(M)
\]

\(M\) is foliated into symplectic leaves and the Casimir functions can be characterized by the property of being constant on each symplectic leaf (equivalently, \(df(x) \in \text{Ker } A(x)\) for any \(x \in M\)).
To each \(A\) we can assign its singular set
\[
S_A = \{x \in M \mid \text{rank } A(x) < \text{rank } A\}
\]
(equivalently, \(S_A\) is the union of all symplectic leaves of non-maximal dimension).
Two Poisson structures \(A\) and \(B\) are compatible if \(\mu A + \lambda B\) is again a Poisson structure.
Let $M$ be a manifold endowed with a linear family $\mathcal{J} = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \mathcal{J}$ are degenerate so that each of them possesses non-trivial Casimir functions.

**Proposition**

Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \mathcal{J}$, then
1) the family of functions

$$\mathcal{F}_\mathcal{J} = \{\text{all Casimir functions of all brackets } A_\mu\}$$

consists of its first integrals;
2) these integrals commute.

Natural questions to discuss: PROPERTIES of $\mathcal{F}_\mathcal{J}$

- Completeness
- Set of critical points
- Equilibrium points
- Non-degeneracy conditions, types
- Codimension one singularities
- Global properties
Euler-Manakov top: $\frac{d}{dt} X = [R(X), X]$, where $R(X)_{ij} = \frac{b_i - b_j}{a_i - a_j} X_{ij}$.

Bi-Hamiltonian structure for the E-M top:
Along with the standard commutator $[X, Y] = XY - YX$ on the space of skew-symmetric matrices, we introduce a new operation $[X, Y]_A = XAY - YAX$ where $A$ is a symmetric matrix.

Observation: E-M top is Hamiltonian w.r.t to the corresponding pencil of compatible Poisson brackets $\{ , \}_{A+\lambda E}$ on $so(n) = so(n)^*$ and, therefore, it admits a large family of commuting integrals of the form

$$\text{Tr} \left( X(A + \lambda E)^{-1} \right)^k$$

which is equivalent to Manakov’s:

$$\mathcal{F}_A = \left\{ \text{Tr} (X + \lambda A)^k \right\}.$$

This family admits a basis that consists of exactly $s = \frac{1}{2} (\dim so(n) + \text{ind} so(n))$ commuting polynomials.
Completeness

Consider a pencil of compatible Poisson brackets \( \mathcal{J} = \{A + \lambda B\} \) on \( M \) and the family of commuting Casimirs \( \mathcal{F}_\mathcal{J} \) as above.

**Question.** Is \( \mathcal{F}_\mathcal{J} \) complete, i.e., sufficient to guarantee complete integrability? How many commuting integrals do we need?

\[
s = \frac{1}{2} (\dim M + \text{corank } \mathcal{J})
\]

Instead of computing the number of independent integrals in \( \mathcal{F}_\mathcal{J} \) it is much better to use the following definition: \( \mathcal{F}_\mathcal{J} \) is **complete** if at a generic point \( x \in M \) the differentials \( df(x), f \in \mathcal{F}_\mathcal{J} \), generate a **maximal** isotropic subspace.

**Theorem**

_The family \( \mathcal{F}_\mathcal{J} \) is complete if and only if at a generic point \( x \in M \) the following condition holds:_

\[
\text{rank } A_\lambda(x) = \text{rank } \mathcal{J} \quad \text{for all } \lambda \in \overline{\mathbb{C}}.
\]

**Codimension two principle.** Let all the brackets \( A_\lambda \), \( \lambda \in \overline{\mathbb{C}} \) have the same rank and \( \text{codim } S_\lambda \geq 2 \) for almost all \( \lambda \in \overline{\mathbb{C}} \). Then \( \mathcal{F}_\mathcal{J} \) is complete.

**Theorem**

_The family of shifts \( \mathcal{F}_a \) is complete on \( \mathfrak{g}^* \) iff \( a \in \mathfrak{g}^* \) is regular and \( \text{codim Sing} > 2 \).
Set of critical points

Suppose that the family of commuting Casimirs $\mathcal{F}_\mathcal{J}$ related to a pencil $\mathcal{J} = \{A + \lambda B\}$ is complete on $M$. However, there are still some singular points $x \in M$ where the commuting functions from $\mathcal{F}_\mathcal{J}$ become dependent:

$$S_\mathcal{J} = \{x \in M \mid \dim D_{\mathcal{F}_\mathcal{J}}(x) < \frac{1}{2}(\dim M + \text{corank } \mathcal{J})\}$$

where $D_{\mathcal{F}_\mathcal{J}}(x) \subset T^*_x M$ is the subspace spanned by the differentials of $f \in \mathcal{F}_\mathcal{J}$.

$S_\mathcal{J}$ is, by definition, the set of critical points of $\mathcal{F}_\mathcal{J}$ (or, equivalently the singular set of the corresponding Lagrangian fibration (see Lecture 1)).

On the other hand, for $\lambda \in \overline{\mathbb{C}}$, we can define the set of “singular points” of $A_\lambda$ in $M$:

$$S_\lambda = \{x \in M \mid \text{rank}(A_\lambda(x)) < \text{rank } \mathcal{J}\}.$$ 

**Theorem**

A point $x$ is critical for $\mathcal{F}_\mathcal{J}$ iff there is $\lambda \in \overline{\mathbb{C}}$ such that $x \in S_\lambda$.

In other words, the set of critical points $S_\mathcal{J}$ of the family $\mathcal{F}_\mathcal{J}$ is the union of ”singular sets” $S_\lambda$ over all $\lambda \in \overline{\mathbb{C}}$:

$$S_\mathcal{J} = \bigcup_{\lambda \in \overline{\mathbb{C}}} S_\lambda$$
Theorem
$x \in M$ is a common equilibrium point for $\mathcal{F}_J$ if and only if the kernels of all generic brackets at this point coincide: $\text{Ker } A_\lambda(x) = \text{Ker } A_\mu(x)$, for all $A_\lambda(x)$ and $A_\mu(x)$ generic.

Theorem II (L. Féher, I. Marshall)
The set of common equilibrium points of $\mathcal{F}_A$ (with $A$ diagonal) is the union of those Cartan subalgebras $\mathfrak{h} \subset \text{so}(n)$ which are common Cartan subalgebras for all commutators $[\ , \ ]_{A+\lambda E}$. One of these Cartan subalgebras is standard:

$$h_0 = \left\{ \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{34} \\ -x_{34} & 0 \end{pmatrix}, \ldots \right\}, \quad x_{i,i+1} \in \mathbb{R}.$$ 

All the others are obtained from $h_0$ by conjugation $h_0 \mapsto P h_0 P^{-1}$ where $P$ is a permutation matrix.
Linearisation of a Poisson structure

According to the splitting theorem (A. Weinstein), locally each Poisson structure $A$ splits into direct product of a non-degenerate Poisson structure $A_{\text{sympl}}$ and the transversal structure $A_{\text{transv}}$ which vanishes at the given point:

$$A = A_{\text{sympl}} \times A_{\text{transv}}$$

The transversal Poisson structure $A_{\text{transv}}$ is well defined and we can consider its linearisation just by taking the linear terms in the Taylor expansion

$$A_{\text{transv}}(x) = \sum c_{ij}^k x_k + \ldots$$

Definition

From the algebraic viewpoint, the linearisation of $A$ at a point $x \in M$ is a Lie algebra $g_A$ defined on $\text{Ker } A(x)$ as follows. Let $\xi, \eta \in \text{Ker } A(x)$ and $f, g$ be smooth functions such that $df(x) = \xi$, $dg(x) = \eta$. Then, by definition,

$$[\xi, \eta] = d\{f, g\}(x) \in \text{Ker } A(x)$$

Remark. If $x \in M$ is a regular point, then $g_A$ is obviously trivial.
$\mathcal{J} = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$. Let us take $x \in M$, fix $\lambda \in \mathbb{C}$ and consider the kernel $\text{Ker} \ A_\lambda(x)$.

On $\text{Ker} \ A_\lambda$ we can introduce two natural structures:
- the Lie algebra $g_\lambda = g_{A_\lambda}$, the linearisation of $A_\lambda$ at the point $x$,
- the restriction of $B$ onto $\text{Ker} \ A_\lambda$.

We can think of them as two Poisson structures on $g_\lambda^*$:
- the first one is linear, i.e., the standard Lie-Poisson structure related to $g_\lambda$,
- the second one is constant $B|_{g_\lambda}$.

**Proposition**
These two Poisson structures are compatible, i.e. generate, a Poisson pencil $\Pi = \Pi(\lambda, x)$.

**Definition**
This Poisson pencil $\Pi$ is called the $\lambda$-linearisation of the pencil $\mathcal{J}$ at $x \in M$. 
Consider two compatible Poisson brackets on a vector space $V$: 
linear $A + \text{constant} \ B$. 

What are “compatibility conditions” for this kind of brackets?

Standard situation is “shift of argument” construction:
The brackets $\{f, g\} (x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$, $\{f, g\}_a (x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ are compatible for each $a = (a_i) \in V$.

Situation can be different:
For $\{f, g\}_A (x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ there may exist constant compatible brackets $\{f, g\}_B (x) = \sum B_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ which are not of the above type. The compatibility condition can be written as

$$B([\xi, \eta], \zeta) + B([\eta, \zeta], \xi) + B([\zeta, \xi], \eta) = 0.$$ 

This identity has a natural cohomological interpretation.

**Remark 1.** If the corresponding Lie algebra is semisimple, then the constant bracket must have the above form $\{ , \}_a$ for some $a \in V$.

**Remark 2.** $\text{Ker} \ B$ is a subalgebra of $\mathfrak{g}$.
Consider two compatible Poisson brackets on a vector space $V$:

linear $A +$ constant $B$

and the corresponding linear-Poisson pencil $\Pi = \{A + \lambda B\}$.

For this pencil $\Pi = \{A + \lambda B\}$ we can construct the family of commuting Casimirs $\mathcal{F}_\Pi$ and ask the question about the structures of singular points. We will say that $\Pi$ is complete, if $\mathcal{F}_\Pi$ is complete.

It is easy to see that $0 \in V$ is a singular point of $\mathcal{F}_\Pi$ and, moreover, it is a common equilibrium.

**Definition**

We say that a complete linear-constant pencil $\Pi = \{A + \lambda B\}$ is *non-degenerate*, if $0 \in V$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$. 
Example

If $A \simeq \mathfrak{so}(3)$ and $B$ is arbitrary, then $\Pi = \{A + \lambda B\}$ is non-degenerate.

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + y^2 + z^2$, $F_2 = ax + by + cz$
Example

$sl(2, \mathbb{R})$–bracket $A$ and constant bracket $B$ defined by an element $\xi \in sl(2, \mathbb{R}) \simeq sl(2, \mathbb{R})^*$:

$$A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $F_1 = x^2 + yz$, $F_2 = ax + by + cz$

Is this pencil non-degenerate?

The answer depends on $\xi$: see next slide
Examples: semisimple case $sl(2, \mathbb{R})$
Examples: semisimple case $sl(2, \mathbb{R})$

Question.
Why are there 3 different cases? How to distinguish them?

Answer.
There are non-trivial elements $\xi \in sl(2, \mathbb{R})$ of three types:

- **elliptic** (eigenvalues are pure imaginary $i\lambda, -i\lambda$);
- **hyperbolic** (eigenvalues are real $\lambda, -\lambda$);
- **nilpotent** (both eigenvalues are zero).

We can distinguish them by using the Killing form:

- elliptic: $(\xi, \xi) < 0$;
- hyperbolic: $(\xi, \xi) > 0$;
- nilpotent: $(\xi, \xi) = 0$.

Equivalently, one may use the sign of $\text{Tr} \xi^2 = -2 \det \xi$ in the standard $2 \times 2$ representation.

Conclusion.
Non-degeneracy $\iff \xi$ is semisimple $\iff \text{Ker } B_\xi$ is a Cartan subalgebra.
Problem.
Describe all “good” Lie algebras \( g \) (equivalently, Lie-Poisson brackets \( A \)) which may produce non-degenerate linear-constant pencils and then for these Lie algebras find necessary and sufficient condition for a constant bracket \( B \) to give indeed a non-degenerate pencil \( \Pi = \Pi(g, B) = \{A + \lambda B\} \).

Such Lie algebras are called non-degenerate too.

Theorem (A. Izosimov)

A linear-constant pencil \( \Pi = \Pi(g, B) \) is non-degenerate (in the complex case) if and only if the Lie algebra \( g \) associated with the linear bracket \( A \) is isomorphic to

\[
\bigoplus \mathfrak{so}(3, \mathbb{C}) \oplus \left( \left( \bigoplus \mathfrak{D} \right) / \mathfrak{h}_0 \right) \oplus \left( \bigoplus \mathbb{C} \right)
\]

where \( \mathfrak{D} \) is the diamond Lie algebra, \( \mathfrak{h}_0 \) is a commutative ideal which belongs to the center of \( \bigoplus \mathfrak{D} \), and \( \text{Ker} \, B \) is a Cartan subalgebra of \( g \).
What is the diamond Lie algebra $\mathcal{D}$?

$\mathcal{D}$ is a four dimensional Lie algebra generated by $x, y, z, u$ with the following relations

\[
[z, x] = y, \quad [z, y] = -x \quad \text{and} \quad [x, y] = u, \quad [u, \mathcal{D}] = 0. \tag{1}
\]

In other words, $\mathcal{D}$ (as a complex Lie algebra) is the non-trivial central extension of $e(2, \mathbb{C})$.

Matrix representation:

\[
\alpha x + \beta y + \theta z + \gamma u \mapsto \begin{pmatrix}
0 & \alpha & \beta & 2\gamma \\
0 & 0 & -\theta & \beta \\
0 & \theta & 0 & -\alpha \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Casimir functions: $F_1 = x^2 + y^2 + 2zu$, $F_2 = u$.

The complex diamond Lie algebra $\mathcal{D}$ has 2 different real forms

- $\mathfrak{g}_{ell}$ defined by (1) and
- $\mathfrak{g}_{hyp}$ defined by $[z, x] = x, [z, y] = -y$, and $[x, y] = u$. 

Theorem (A. Izosimov)

A real Lie algebra \( \mathfrak{g} \) is non-degenerate iff

\[
\mathfrak{g} \cong \left( \bigoplus \mathfrak{so}(3, \mathbb{R}) \right) \oplus \left( \bigoplus \mathfrak{sl}(2, \mathbb{R}) \right) \oplus \left( \bigoplus \mathfrak{so}(3, \mathbb{C}) \right) \oplus 

\left( \left( \left( \bigoplus \mathfrak{g}_{\text{ell}} \right) \oplus \left( \bigoplus \mathfrak{g}_{\text{hyper}} \right) \oplus \left( \bigoplus \mathfrak{g}_{\text{foc}} \right) \right) / \mathfrak{h}_0 \right) \oplus \left( \bigoplus \mathbb{R} \right)
\]

where

- \( \mathfrak{g}_{\text{ell}} \) and \( \mathfrak{g}_{\text{hyp}} \) are the non-trivial central extensions of \( e(2) \) and \( e(1, 1) \) (equivalently, they are real forms of \( \mathfrak{D} \)),
- \( \mathfrak{g}_{\text{foc}} = \mathfrak{D} \) treated as real Lie algebra,
- \( \mathfrak{h}_0 \) is a commutative ideal which belongs to the center.

A linear-constant pencil \( \Pi(\mathfrak{g}, B) \) is non-degenerate if \( \mathfrak{g} \) is non-degenerate and \( \text{Ker} \, B \) is a Cartan subalgebra of \( \mathfrak{g} \).

The type of the singularity is naturally defined by the “number” of elliptic, hyperbolic and focus components in the above decomposition.
Now let $J = \{A + \lambda B\}$ be an arbitrary pencil of compatible Poisson brackets. We consider the commutative family of functions $\mathcal{F}_J$ and a singular point $x \in S_J$.
This means, that at this point there are non-trivial characteristic numbers $\lambda_i$ for the pencil $J(x) = \{A(x) + \lambda B(x)\}$.
For each of them we can consider the $\lambda_i$-linearisation. Is $x$ non-degenerate?

Theorem (A. Izosimov)

Let $J = \{A + \lambda B\}$ be a pencil of compatible Poisson brackets, $\mathcal{F}_J$ be the corresponding family of commuting Casimirs and $x \in M$ singular point for $\mathcal{F}_J$.
This point is non-degenerate if and only if for every characteristic number $\lambda_i$,
1. the $\lambda_i$-linearisation of $J$ at $x$ is non-degenerate;
2. the pencil $J(x) = \{A + \lambda B\}$ is diagonalisable (i.e. all the Jordan blocks are $2 \times 2$).
Non-degeneracy of equilibria

**Definition**
A common equilibrium point \( x \in M \) of commuting Hamiltonians \( f_1, \ldots, f_n \) is called **non-degenerate** if their “quadratic parts” parts \( d^2 f_1(x), \ldots, d^2 f_n(x) \) generate a Cartan subalgebra in \( sp(T_x M, \omega) \).

**Theorem**

Let \( x \) be a common equilibrium point for \( F_J \). Suppose that the characteristic numbers of \( A_\lambda(x) = A(x) + \lambda B(x) \) all have multiplicity 2, and there exists \( f \in F_J \) such that the corresponding linearization operator \( A_f : T_x O \rightarrow T_x O \) is non-degenerate. Then \( x \) is non-degenerate.

**Theorem II**

Let \( X \) be a \( 2 \times 2 \) block-diagonal skew-symmetric matrix (as above). For each pair \( x_{i,i+1}, x_{j,j+1} \), consider the two roots \( \lambda_{ij}, \lambda'_{ij} \) of the equation

\[
\frac{x_{i,i+1}^2}{x_{j,j+1}^2} = \frac{(a_i + \lambda)(a_{i+1} + \lambda)}{(a_j + \lambda)(a_{j+1} + \lambda)}.
\]

If \( \lambda_{ij}, \lambda'_{ij} \) \( (i \neq j, \ i, j = 1, 3, \ldots, 2n - 1 \) are all distinct, then \( X \) is a non-degenerate equilibrium point for \( F_A \).
General criterion. Step 2: Linearisation of a Poisson pencil

\[ \mathcal{J} = \{ A_\lambda = A + \lambda B \} \] is a pencil of compatible Poisson brackets and \( x \in M \).

Let us take \( x \in M \), fix \( \lambda \neq 0 \) and consider the kernel \( \text{Ker} \ A_\lambda(x) \).

On \( \text{Ker} \ A_\lambda \) we can introduce two natural structures:

- the Lie algebra \( g_\lambda = g_{A_\lambda} \), the linearisation of \( A_\lambda \) at the point \( x \),
- the restriction of \( A \) onto \( \text{Ker} \ A_\lambda \).

We can think of them as two Poisson structures on \( g_\lambda^* \):

- the first one is linear, i.e., the standard Lie-Poisson structure related to \( g_\lambda \),
- the second one is constant \( A|_{g_\lambda} \).

Proposition

These two Poisson structures are compatible, i.e. generate, a Poisson pencil
\[ \Pi = \Pi(\lambda, x) \].

Definition

This Poisson pencil \( \Pi \) is called the \( \lambda \)-linearisation of the pencil \( \mathcal{J} \) at \( x \in M \).
Consider two compatible Poisson brackets on a vector space $V$:

\[
\text{linear } A + \text{ constant } B.
\]

**Example (Argument shift method)**

The brackets $\{f, g\}(x) = \sum c^k_{ij} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$, $\{f, g\}_a(x) = \sum c^k_{ij} a_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ are compatible for each $a = (a_i) \in V$. If the corresponding Lie algebra is semisimple, then the constant bracket must have the above form for some $a \in V$.

For this special kind of Poisson pencils $\Pi = \{A + \lambda B\}$ we can construct the family of commuting functions $\mathcal{F}_J$ and ask the question about the structures of singular points. We will say that $J$ is complete, if $\mathcal{F}_J$ is complete.

**Definition**

We say that a complete linear-constant pencil $\Pi = \{A + \lambda B\}$ is **non-degenerate**, if $0 \in V$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$.

**Example**

If $A \simeq so(3)$ and $B$ is arbitrary, then $\Pi = \{A + \lambda B\}$ is non-degenerate.
Theorem (A. Izosimov)

Let \( J = \{ A + \lambda B \} \) be a pencil of compatible Poisson brackets, \( \mathcal{F}_J \) be the associated commutative family of functions and \( x \in M \) singular point for \( \mathcal{F}_J \). This point is non-degenerate if and only if for every characteristic number \( \lambda_i \),

1. the \( \lambda_i \)-linearisation of \( J \) at \( x \) is non-degenerate;
2. the corank of the \( \lambda_i \)-linearisation equals to \( \text{corank} J \).

Problem. Is it possible to classify non-degenerate pencils?

Theorem (A. Izosimov)

A linear-constant pencil \( \Pi = \{ A + \lambda B \} \) is non-degenerate (in the complex case) if and only if the Lie algebra \( \mathfrak{g} \) associated with the linear bracket \( A \) is isomorphic to

\[
\bigoplus \text{so}(3, \mathbb{C}) \oplus \left( \bigoplus \mathfrak{D} / \mathfrak{h}_0 \right) \oplus \bigoplus \mathbb{C}
\]

where \( \mathfrak{D} \) is the diamond Lie algebra, \( \mathfrak{h}_0 \) is a commutative ideal which belongs to the center of \( \bigoplus \mathfrak{D} \), and \( \text{Ker } B \) is a Cartan subalgebra of \( \mathfrak{g} \).
A new example: Rubanovskii case

The **Rubanovskii case** is an integrable generalization of the classical Steklov-Lyapunov case obtain by adding "gyrostatic terms". For our purposes the only important thing is

**Proposition (follows from the Lax pair discovered by Yu. Fedorov)**

The Rubanovskii system is Hamiltonian w.r.t. the pencil generated by the following compatible Poisson brackets:

\[
\Pi_0 = \begin{pmatrix}
0 & z_3 - b_3 p_3 & -z_2 + b_2 p_2 & 0 & p_3 & -p_2 \\
-z_3 + b_3 p_3 & 0 & z_1 - b_1 p_1 & -p_3 & 0 & p_1 \\
z_2 - b_2 p_2 & -z_1 + b_1 p_1 & 0 & p_2 & -p_1 & 0 \\
0 & p_3 & -p_2 & 0 & 0 & 0 \\
-p_3 & 0 & p_1 & 0 & 0 & 0 \\
p_2 & -p_1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Pi_1 = \begin{pmatrix}
0 & b_3 z_3 - g_3 & -b_2 z_2 + g_2 & 0 & 0 & 0 \\
-b_3 z_3 + g_3 & 0 & b_1 z_1 - g_1 & 0 & 0 & 0 \\
b_2 z_2 - g_2 & -b_1 z_1 + g_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_3 & -p_2 \\
0 & 0 & 0 & -p_3 & 0 & p_1 \\
0 & 0 & 0 & p_2 & -p_1 & 0
\end{pmatrix}
\]

\[z, p\] are coordinates in the phase space \(\mathbb{R}^6\), \(b\) and \(g\) are geometric parameters.
A new example: Rubanovskii case

The algebraic structure of $\Pi_1 - \lambda \Pi_0$ becomes clear if we change variables:

$$\tilde{z}_i = z_i + \lambda p_i + \frac{g_i}{\lambda - b_i}, \quad p_i \text{'s remain the same}$$

Then:

$$\Pi_1 - \lambda \Pi_0 = \begin{pmatrix}
0 & (b_3 - \lambda)\tilde{z}_3 & -(b_2 - \lambda)\tilde{z}_2 \\
-(b_3 - \lambda)\tilde{z}_3 & 0 & (b_1 - \lambda)\tilde{z}_1 \\
(b_2 - \lambda)\tilde{z}_2 & -(b_1 - \lambda)\tilde{z}_1 & 0 \\
0 & p_3 & -p_2 \\
-p_3 & 0 & p_1 \\
p_2 & -p_1 & 0
\end{pmatrix}$$

Thus, $\Pi_1 - \lambda \Pi_0$ splits into the direct sum of two brackets, one of which is the standard $so(3)$-bracket and the other is isomorphic to either to $so(3)$, or to $sl(2)$ depending on the signs of $b_i - \lambda$, $i = 1, 2, 3$.

**Question:** What is the singular set?

**Answer:** Those points where the rank of $\Pi_1 - \lambda \Pi_0$ drops.
A new example: Rubanovskii case

Theorem
A point \((z, p)\) belongs to the critical set iff there is \(\lambda \in \mathbb{C} \setminus \{b_1, b_2, b_3\}\) such that

\[
z_i + \lambda p_i + \frac{g_i}{\lambda - b_i} = 0, \quad i = 1, 2, 3.
\]

By applying in a similar way this bi-Hamiltonian approach, we immediately obtain some further results

Theorem
A point \((z, p)\) is a common equilibrium iff

\[
\text{rank} \begin{pmatrix} p_1 & z_1 - b_1 p_1 & g_1 - b_1 z_1 \\ p_2 & z_2 - b_2 p_2 & g_2 - b_2 z_2 \\ p_3 & z_3 - b_3 p_3 & g_3 - b_3 z_3 \end{pmatrix} = 1.
\]

Theorem
Let \(\gamma\) be a critical closed trajectory passing through \((z, p)\) with parameter \(\lambda\). Then \(\gamma\) is non-degenerate iff

\[
C = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3) \sum_{i=1}^{3} \left( (\lambda - b_i)p_i - \frac{g_i}{\lambda - b_i} \right)^2 \frac{1}{\lambda - b_i} \neq 0
\]

Moreover, if this expression \(C > 0\) then \(\gamma\) is stable, and if \(C < 0\) then \(\gamma\) is unstable.
Euler-Manakov tops on $so(n)$

**Theorem (L. Féher, I. Marshall)**

The set of common equilibrium points of $\mathcal{F}_A$ (with $A$ diagonal) is the union of those Cartan subalgebras $\mathfrak{h} \subset so(n)$ which are common Cartan subalgebras for all commutators $[\ , \ ]_{A+\lambda E}$. One of these Cartan subalgebras is standard:

$$
\mathfrak{h}_0 = \left\{ \begin{pmatrix}
0 & x_{12} \\
-x_{12} & 0 \\
 & & & & & & x_{34} \\
 & & & & & -x_{34} & 0 \\
 & & & & & & & \ddots
\end{pmatrix}, \ x_{i,i+1} \in \mathbb{R} \right\}.
$$

All the others are obtained from $\mathfrak{h}_0$ by conjugation $\mathfrak{h}_0 \mapsto P\mathfrak{h}_0 P^{-1}$ where $P$ is a permutation matrix.

**Theorem (Oshemkov, AB)**

Let $X$ be a $2 \times 2$ block-diagonal skew-symmetric matrix (as above). For each pair $x_{i,i+1}, x_{j,j+1}$, consider the two roots $\lambda_{ij}, \lambda'_{ij}$ of the equation

$$
\frac{x_{i,i+1}^2}{x_{j,j+1}^2} = \frac{(a_i + \lambda)(a_{i+1} + \lambda)}{(a_j + \lambda)(a_{j+1} + \lambda)}.
$$

If $\lambda_{ij}, \lambda'_{ij}$ ($i \neq j$, $i, j = 1, 3, \ldots, 2n-1$) are all distinct, then $X$ is a non-degenerate equilibrium point for $\mathcal{F}_A$. 
Euler-Manakov tops on $so(n)$

For each pair of blocks
\[
\begin{pmatrix}
0 & x_{i,i+1} \\
-x_{i,i+1} & 0
\end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]
consider the function $f(x) = \frac{(x-\lambda_1^2)(x-\lambda_2^2)}{\omega^2(\lambda_1+\lambda_2)^2}$. and let $f(\infty) = \frac{1}{\omega^2(\lambda_1+\lambda_2)^2}$.
By drawing the graphs of all these functions on the same plane $\mathbb{R}^2$, we obtain a collection of parabolas called the \textit{parabolic diagram} $\mathcal{P}$. For simplicity we assume that $n$ is even.
We say that this diagram is generic if any two parabolas intersect exactly at two points (including complex intersections and intersections at infinity)

**Theorem (A. Izosimov)**

- The equilibrium point is non-degenerate iff the parabolic diagram $\mathcal{P}$ is generic:
  - each intersection point in the upper half plane corresponds to an elliptic component;
  - each intersection point in the lower half plane corresponds to a hyperbolic component;
  - each complex intersection corresponds to a focus component.
- If $\mathcal{P}$ is generic, all intersections are real and located in the upper half plane, then the equilibrium is stable.
- If there is either a complex intersection or an intersection point in the lower half plane, then the equilibrium point is unstable.